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REPORT NO. 675

Investigation of the Propagation of Blast Waves Over Relatively Large Distances and the Damaging Possibilities of Such Propagation

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BALLISTIC RESEARCH LABORATORIES

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(6) Investigation of the Propagation of Blast Waves Over Relatively Large Distances and the Damaging Possibilities of Such Propagation

(10) WARREN W./BERNING

(16) ORD

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BALLISTIC RESEARCH LABORATORIES

REPORT NO. 675

W. Berning
Aberdeen Proving Ground, Md.
8 July 1948

INVESTIGATION OF THE PROPAGATION OF BLAST WAVES OVER RELATIVELY LARGE DISTANCES AND THE DAMAGING POSSIBILITIES OF SUCH PROPAGATION

ABSTRACT

The test firing of large guns and explosives has resulted, on numerous occasions, in complaints from inhabitants residing outside the testing area. The size of the testing area should preclude any such damaging effects but the presence of these effects indicated that on certain days a blast wave is propagated to unusually great distances with a relatively high pressure difference across the wave surface. Consideration of seismic waves and air waves led to the conclusion that this unusual blast effect is propagated through the atmosphere and hence is, to a great extent, dependent upon existant meteorological conditions.

In this paper an attempt is made to determine those meteorological conditions which are conducive to unusual blast wave propagation and to establish a few rules of thumb which will enable a reasoning person, with the aid of temperature and wind soundings of the lower atmosphere, to determine the presence of dangerous propagating conditions. These rules of thumb are not based upon a rigid theoretical analysis simply because our lack of knowledge of microscopic meteorological phenomena and the physical characteristics of spherical blast waves would make such a rigid analysis meaningless.

The work of various individuals in the field of sound propagation through the atmosphere is studied and the results of their work are employed in the analysis of the subject problem. While initially a blast wave may differ greatly from an ordinary sound wave, a short distance from the explosive source these two types of waves travel with identical velocities and are affected in equal degrees by existing meteorological conditions.

Because of mathematical and physical difficulties the necessary approach to the problem was to study certain ideal atmospheric structures and to determine the altitude function of temperature and wind (and relative humidity) which is necessary if destructive blast propagation is to take place. The results of these studies indicated that a necessary, though not sufficient condition is that a temperature inversion exists in the atmosphere through which the blast wave is propagated; i.e., an increase in temperature with altitude.

The necessary increase in temperature with an increase in altitude to produce destructive blast propagation immediately suggests certain meteorologic phenomena associated with this temperature condition. Employing these meteorologic phenomena enables one to set forth certain rules of thumb which will indicate dangerous blast conditions.

1. A high pressure area (atmospheric) which is stationary over the area in question becomes dangerous after the first day of its presence.

2. Large diurnal temperature variations at the earth's surface indicate unfavorable firing conditions.

3. Poor visibilities and light winds in the early morning followed by cloudless conditions throughout the day indicate dangerous blast propagation.

4. Light surface winds and low relative humidity at the ground plus the presence of stratified cloud deck below 10,000 feet indicate dangerous firing conditions.

Poor visibilities and light winds in the early morning followed by the formation of cumulus (swelling) clouds in the late morning and early afternoon indicate safe conditions for an afternoon firing.

Several situations in which destructive blast was reported were picked at random and the meteorological conditions occurring at that time were examined. It was found in each of these cases that a positive vertical temperature gradient occurred in the lower atmosphere (below 10,000 feet) and that the weather conditions fit one or more of the 4 situations described above.

INTRODUCTION

An integral portion of the research carried on at the Aberdeen Proving Ground involves the study of statically detonated charges. Since some of these charges must of necessity be quite large, the blast waves produced by the explosions have large pressure discontinuities across their boundries and are capable of inflicting serious damage upon objects a short distance from the pressure wave source. However, beyond a relatively short distance, the intensity of the pressure or blast wave decreases approximately as the square of the distance from the source, and the area of the testing range is such that, theoretically, no blast damage should occur to objects without the confines of this area. Unfortunately, blast damage has occured in the past — at considerable expense to the Government — and it seems highly desirable to investigate the conditions under which this destructive blast is propagated. During the war the press and importance of blast investigations — of charges dropped from aircraft as well as static detonations — forbade consideration of minor damage to private property and the policy of firing first and paying afterwards was generally adopted. With the cessation of hostilities came budget reductions and it became at once desirable to find some method of predicting, beforehand, the presence of unfavorable meteorological conditions, i.e., conditions when blast damage is likely to occur.

The phenomenon of unusual sound or blast propagation has, of course, been known for many years. It has been the subject of many investigations during the past century by such men as Stokes (1), Reynolds (2), Milne (3), and Whipple (4) in England and Gutenberg (5) and Duckert (6) in Germany. Successive zones of audibility and silence occuring when severe explosions have taken place have been plotted and analyzed and their causes determined. During the first World War the sound ranging of artillery fire was used for determining enemy positions and it was quickly noted that on occasions the intensity of recorded gunfire was quite great.

During the months of November and December of the year 1940, the US Bureau of Mines (7) conducted a series of experiments at the Aberdeen Proving Ground in which seismic and air pressure measurements were made in coincidence with the firing of relatively low calibre guns. For the experiments several houses of moderate size and construction located on the northwesterly boundry of the Proving Ground were chosen. Seismometers were mounted on the floors of these dwellings and measurements taken while firing was in progress. Results of the experiments indicated that the firing of 8 inch guns and 8 inch Howitzers produced no superficial damage to these houses located only 1 to 2 miles distant from the point of fire. No further experiments along these lines were conducted by the Bureau of Mines.

In June, 1944, R. G. Sachs (8) of the Ballistic Research Laboratories published a report entitled, "The Limiting Distances for Superficial Damage to Buildings due to the Firing of Guns and High Explosives." In this report the author reviewed the old Bureau of Mines' experiments in the light of the complaints then being received from inhabitants residing near the Proving Ground. A survey of these complaints and claims against the Government for damage caused by firing programs enabled Sachs to evolve an empirical formula which defined the limiting distance beyond which destructive blast from detonated charges evidently did not occur. This formula is given as

$$R = 3750 \sqrt[3]{W}$$

?

R = Limiting destructive range (feet)
W = Weight of explosive (pounds)
= 1/7 weight of propellant charge if gun is used.

Re-examination of the data gathered in the Bureau of Mines' experiments revealed that in light of the above formula no destructive effects from the gunfire could be expected. The experiments did show, however, that the transmission of shock through the ground was of no more importance than the transmission of blast through the atmosphere.

The ever present danger of destructive blast propagation from detonated charges resulted, quite naturally, in the formation of a central uthoritative source which had the unpleasant task of determining whether or not a firing should take place i.e., whether or not destructive blast was likely to occur. This authoritative source, the Safety Officer, could only base his judgments on analogous situations when unusual blast propagation had been known to occur, and, because the responsibility was great and the criteria meager, much time and money have been wasted by overcautious delays in firing programs. Actually it is not at all unreasonable to suppose that more government funds have been expended in firing delays than would have been used had the firings proceeded as scheduled and payments made for any reasonable claims for resulting damage. It most certainly is a fact that complaints have been received (although no monetary claims were involved) for a firing which was believed "safe".

It must be pointed out that no reflection upon the ability of any individual is implied in these statements. The basis of prediction, built upon the weakest of foundations, is, of course, at fault.

The investigation of unusual blast propagation through the atmosphere is detailed in the following pages and some attempts are made to establish certain fundamental rules for producting a more accurate prediction of dangerous propagating conditions.

THEORY

The propagation of sound through the atmosphere is a well understood physical phenomenon and is quite capable of being mathematically analyzed. The wave and wave front characteristics of a small pressure disturbance immediately suggest a Huygen's construction for determining future frontal positions of such a disturbance. Vertical or horizontal wind and temperature gradients which exist in the atmosphere may be simply treated by considering the propagation velocity to vary along the wave surface of a small pressure disturbance and succeeding positions of the wave front plotted accordingly. Actually, in many respects, phenomena in sound propagation through the atmosphere are quite analogous to the better known and better understood optical counterparts. Refraction, reflection, and diffraction are common occurrences in either type of propagation and their causes may be readily determined by an examination of prevailing meteorological conditions. The analogy ceases, however, when consideration is taken of the fact that density differences rather than temperature variations affect the passage of light through the atmosphere. Thus in a normal atmospheric temperature structure (temperature decreasing with altitude) light will be refracted downward toward the region of greater density whereas sound will be refracted upward toward the region of lower propagation velocity. Also wind gradients (absolute values) have a negligible effect upon light propagation but are most important in the treatment of sound wave motions.

Lord Rayleigh (9) has furnished a most useful tool for the study of sound propagation; namely, the concept of "rays of sound." Although the concept may be physically artificial, the location of sound fronts by suitable velocity vectors (sound rays) and Huygen's constructions is most accurate. In fact, the propagation of sound through a medium in which the velocity vectors change continuously — but not according to theoretical or empirical laws — may only be treated accurately by just such a concept. Figures 1a and

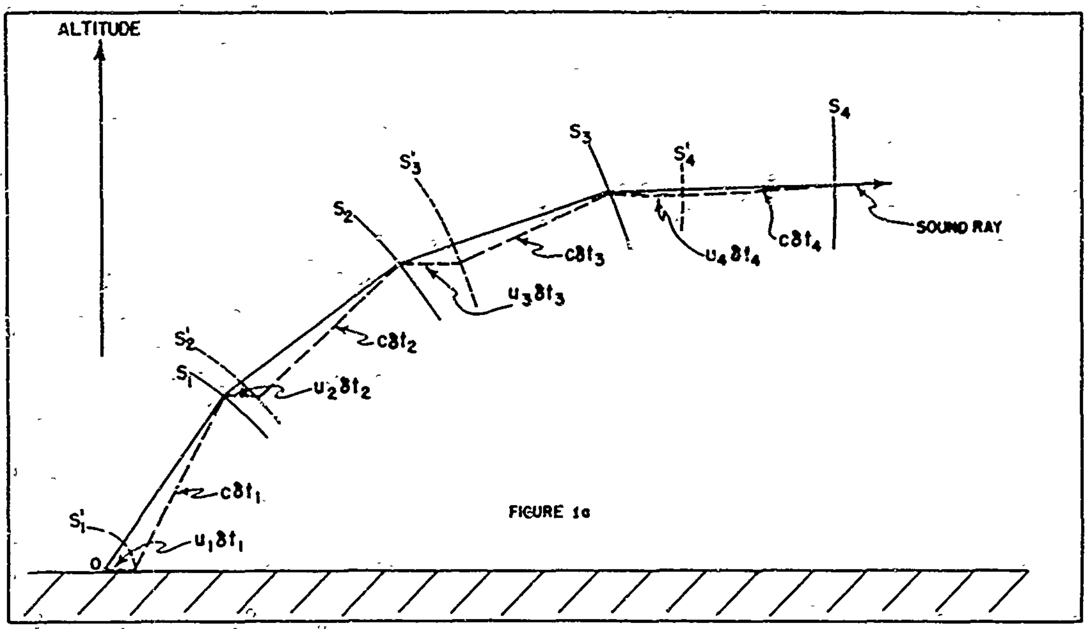


FIGURE 1a. Shows the situation giving rise to refraction of sound by wind effects alone. u_1, u_2, u_3 and u_4 are the varying wind velocities. C is the velocity of sound arising from a uniform temperature in the medium. S_1, S_2, S_3 and S_4 are sound wave frontal positions after time intervals of δt_1, δt_2, δt_3 and δt_4. Figure 1a shows quite obviously that the sound ray is the resultant of the vectors of $u\ \delta t$ and $c\ \delta t$ and also that it is not normal to the wave front in the general case. In the atmosphere where δt must of necessity be taken small, the sound ray would appear as a continuous curve.

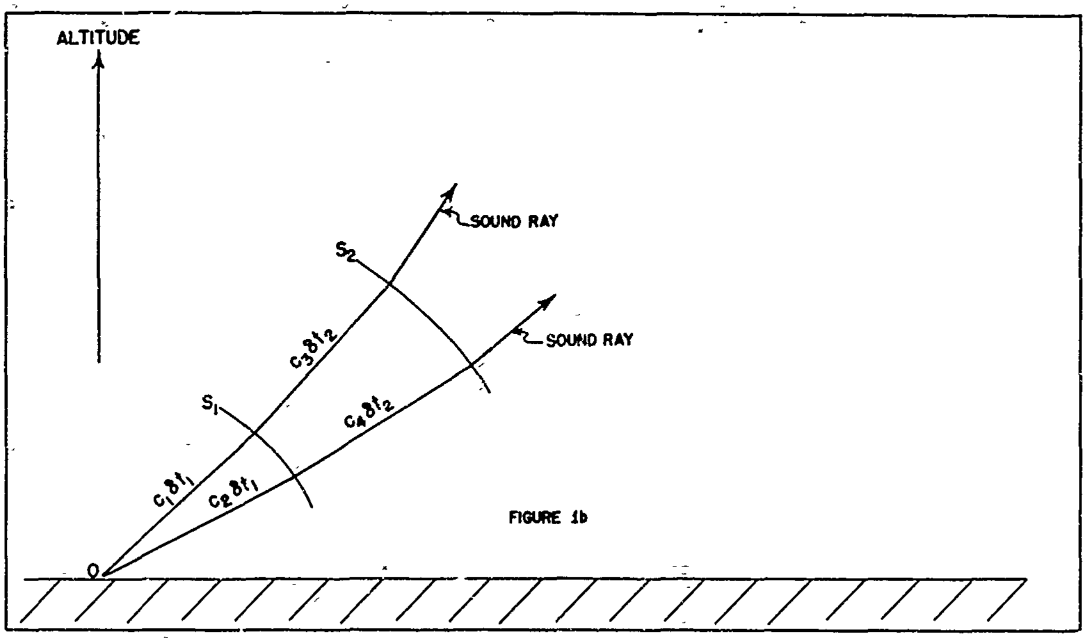


FIGURE 1b. Shows the refraction of sound by a temperature gradient alone. In this case there is no wind. Since, for the purpose of illustration, δt had to be made finitely large, the sound rays are drawn as straight lines and the sound velocities c_1, c_2, c_3 and c_4 must be taken as average velocities over the altitude interval indicated. In the atmosphere, of course, the sound rays would appear as continuous curves since δt could be made sufficiently small.

1b illustrate the relationship between sound ray and sound front under different propagating conditions.

Thus far the discussion has been concerned only with the propagation of sound through the atmosphere whereas the problem itself involves the propagation of blast or shock waves. Because a shock wave is characterized by a nearly instantaneous pressure rise along its frontal surface and because the excess pressure behind the shock surface may even be whole multiples of the atmospheric pressure (close to the origin of the shock wave), the atmosphere is heated irreversably (entropy increase) by the passage of such a wave. For these reasons — and also because of the existence of individual atmospheric particle motions due to the detonation — the blast disturbance does not proceed outward at the velocity of sound but rather at a velocity dependent upon a) the size of the detonated charge, b) the distance of the blast disturbance from the activating source and c) the existing meteorological conditions. However, since the total energy of a blast wave must remain constant as the wave proceeds from the source, the pressure behind the shock surface decreases and eventually reaches a value comparable to that found in ordinary sound waves. Thereafter the blast wave is propagated at the local velocity of sound.

The rapidity with which the pressure decreases (as a function of distance from the source) behind a shock surface is dependent, quite obviously, upon the size of the originating disturbance and seems to be best determined by empirical methods. Taylor (10) has discussed the behavior of a blast wave emanating from the detonation of a 500 lb. charge and gives tables showing the variation of pressure behind the shock wave as a function of distance from the source. Extrapolation of Taylor's results (Fig. 2) shows that, at 500 feet, the velocity of the blast wave (from a 500 lb. bomb) is less than 1.01 times the velocity of sound at the same atmospheric temperature. Over distances of two miles or greater it would be extremely difficult to distinguish time differences of arrival between sound and blast waves originating from a common point at a common time. Even for charges of considerably greater magnitude, propagation velocity of the blast wave may be assumed equal to that of sound where distances involved are larger than two or three miles. This assumption, while not entirely valid, enables one to apply the propagation laws of sound to blast wave phenomenon — about which far less is known and understood (referring specifically to the spherical case).

One further difference between sound and blast phenomena must be indicated at this point. The wave length of a blast wave changes as a function of distance from the source (at least during the relatively early expansion) and the attenuation suffered by a pressure wave in the atmosphere seems to be dependent, to a large extent, upon the frequency of the disturbance. Unfortunately very little is known about this change in wave length mentioned above and further assumptions must be made. Since sound waves do not vary in frequency along a path it seems logical to assume that when the pressure amplitude of a blast wave approaches that found in a sound disturbance the wave length tends to become constant and undergoes no change beyond a certain limiting distance. If this assumption is made it then follows that blast waves originating from different size charges will have different final frequencies (other things being equal) and will be affected in different ways during passage through the atmosphere. Data on sound ranging of guns during the first World War (11) indicated that the wave length of the pressure wave from gun-blast was proportional to the calibre of the gun and, for a 15 inch gun the frequency, at a great distance from the source, was found to be 4 or 5 cycles per second. (Wave length defined as time length of positive pressure phase).

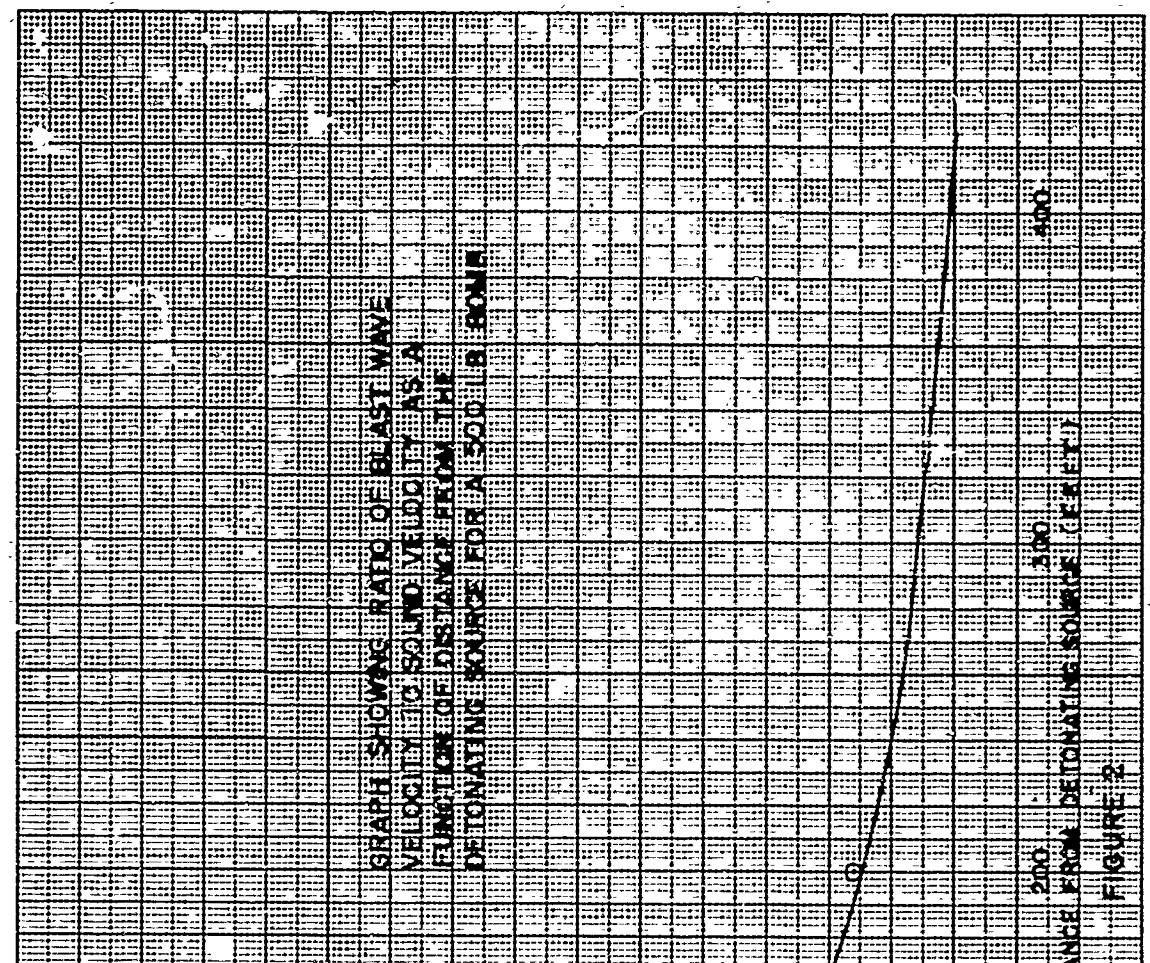

GRAPH SHOWING RATIO OF BLAST WAVE
VELOCITY TO SOUND VELOCITY AS A
FUNCTION OF DISTANCE FROM THE
DETONATING SOURCE FOR A 500 LB. BOMB
U/a - RATIO OF WAVE VELOCITY TO SOUND VELOCITY
1.15
1.10
1.05
1.00
0
100
200
300
400
X = DISTANCE FROM DETONATING SOURCE (FEET)

FIGURE 2

Consideration of the facts and suppositions indicated above leads to the following conclusions:

1) Blast waves may be treated as ordinary sound waves, with regard to propagation and physical characteristics, beyond a relatively short distance from the source of disturbance.

2) Propagation laws developed for sound phenomena do not consider attenuation as a function of frequency or as a function of water vapor and dust particles in the air. Therefore, a disturbance of greater wave length, such as a blast wave, may be expected to conform more closely to the idealized formulae than one at the lower or intermediate audible levels.

In 1916, Fujiwhara (12) published the second of two reports in which he described an investigation of the sound ray paths emanating from the Asama volcano eruptions of 1909 to 1913. In treating this problem Fujiwhara was forced to make two basic and simplifying assumptions:

a) only wind and air temperature affected propagation velocities and

b) horizontal gradients in wind and temperature were zero (i.e. wind and temperature varied only with height). Thus if x and y are rectangular coordinates of the earth's surface considered as a plane, the projected curve of a sound ray on this plane is given by

II. 1 $$\pm x = \int_0^z \frac{c^2 \cos\psi + u_1 \left\{ (c_o/\sin i_o) - (u_1 - u_o)\cos\psi - (v_1 - v_o)\sin\psi \right\} dz}{c\left[\left\{ c_o/\sin i_o) - (u_1 - u_o)\cos\psi - (v_1 - v_o)\sin\psi \right\}^2 - c^2 \right]^{1/2}},$$

II. 2 $$\pm y = \int_0^z \frac{c^2 \sin\psi + v_1 \left\{ (c_o/\sin i_o) - (u_1 - u_o)\cos\psi - (v_1 - v_o)\sin\psi \right\} dz}{c\left[\left\{ (c_o/\sin i_o) - (u_1 - u_o)\cos\psi - (v_1 - v_o)\sin\psi \right\}^2 - c^2 \right]^{1/2}},$$

where

z = altitude - vertical axis;

c = velocity of sound due to temperature alone. (Equation A.7 Appendix A)

u, v = horizontal wind components in x and y directions respectively.

i = angle between z-axis and normal to the wave front.

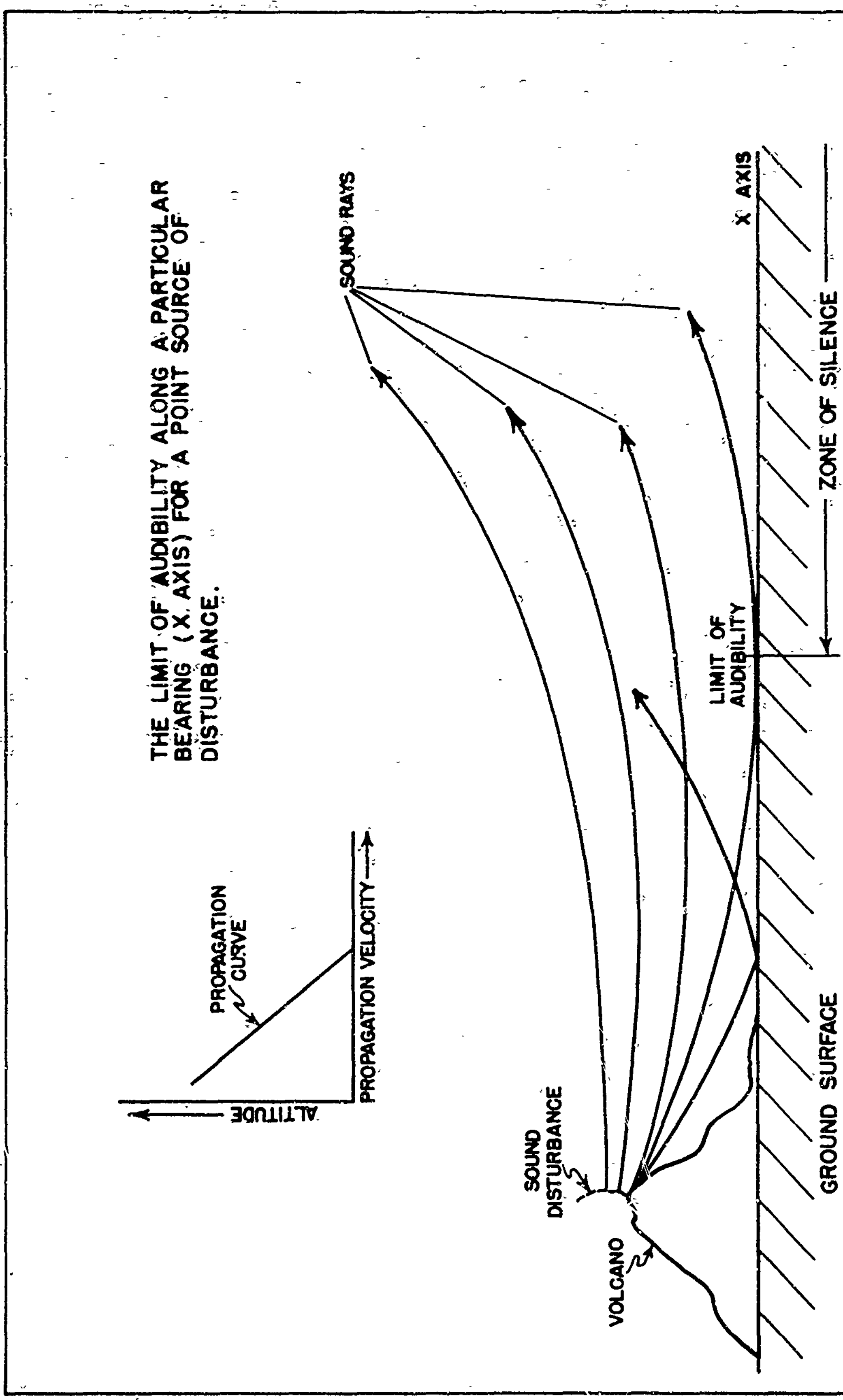
ψ = azimuth of wave front normal for sound ray in question with respect to x-axis.

Subscripts 0 and 1 refer to initial conditions and conditions at end of altitude interval dz.

In actually applying the above equations Fujiwhara was forced to make several further simplifications and assumptions.

1. The x-axis was chosen along a number of bearings in succession so that the wind effect v sin ψ was negligible. This determined a maximum and minimum x and defined the limits of audibility (See Figure 3).

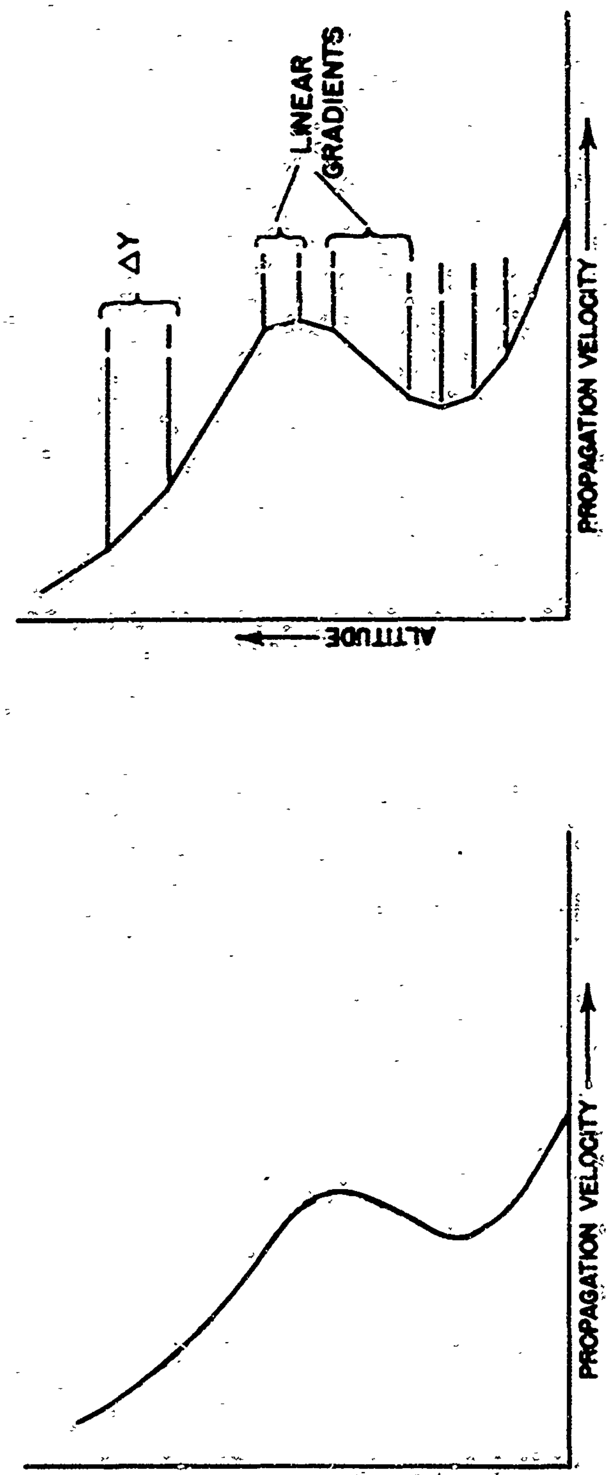
2. For the observed atmospheric structure an artificial one was substituted made up of finite altitude intervals in which the temperature and wind gradients were linear (see Figure 4).



THE LIMIT OF AUDIBILITY ALONG A PARTICULAR BEARING (X AXIS) FOR A POINT SOURCE OF DISTURBANCE.

FIGURE 3

SIMPLIFIED ATMOSPHERIC STRUCTURE TO BE USED
IN COMPUTING SOUND RAY PATHS BY FUJIWHARA'S METHOD



SIMPLIFIED PROPAGATION CURVE FOR USE
IN COMPUTATIONS

TRUE PROPAGATION CURVE

FIGURE 4

3. A separate calculation was carried out for the translational movement of the entire air mass (consequently the sound ray as well) for the period during which the sound traversed the air mass.*

In 1921 E. A. Milne (3) published a theoretical treatise on the propagation of sound through the atmosphere. By considering the equations of motion of a sound ray,

II. 3
$$\frac{dx}{dt} = lc + u, \quad \frac{dy}{dt} = mc + v, \quad \frac{dz}{dt} = nc + w$$

where: l, m and n are direction cosines of normal to wave surface at (x,y,z); u, v , w are wind components in x, y and z directions respectively; c is the velocity of sound due to temperature effects alone. Milne was able to derive the general solution for the path of a sound ray through the atmosphere (see Appendix B). This solution is given as

II. 4
$$\frac{1}{l}\left(\frac{dl}{dt} + \frac{\partial c}{\partial x} + l\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial x} + n\frac{\partial w}{\partial x}\right)$$
$$= \frac{1}{m}\left(\frac{dm}{dt} + \frac{\partial c}{\partial y} + l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y} + n\frac{\partial w}{\partial y}\right)$$
$$= \frac{1}{n}\left(\frac{dn}{dt} + \frac{\partial c}{\partial z} + l\frac{\partial u}{\partial z} + m\frac{\partial v}{\partial z} + n\frac{\partial w}{\partial z}\right)$$

In addition Milne was able to show that by assuming a stratified atmosphere (i.e. zero horizontal wind and temperature gradients)

II. 5
$$\frac{l}{m} = \text{constant.}$$

Equation II. 5 is most important because it states mathematically that in a stratified medium the normals to the wave front for any particular sound ray remain parallel to a fixed plane which is itself perpendicular to the planes of stratification.

By choosing the x-axis (equations II. 4) in such a manner that, for a particular ray, the m direction cosine is equal to zero, equations II. 4 may be separated and integrated to give the following result:

*This translational movement is separate from the refracting effect of the wind since the former is a motion relative of the fixed point of disturbance while the latter effect arises from the presence of a vertical wind gradient.

II. 6 $$c \sec \theta + u = c_o \sec \theta + u_o = \text{constant},$$

where θ = angle between sound ray normal and the x-axis.

Equation II. 6 is the mathematical expression for the "Law of Sound Refraction."

One more investigation of importance which has appeared recently, (13) details the work of P. Rothwell and associates in England in attempting a simple integration of the equations of motion of a sound ray in the atmosphere. The assumptions used in the analysis were very similar to those made by Fujiwhara and Milne, namely:

1. The sound ray path was constrained to be entirely within a vertical plane.
2. The atmosphere was considered to be stratified in horizontal layers (no vertical wind currents) and the observed structure could be broken into layers where the wind and temperature gradients were linear (see Figure 4).

Considering a small altitude interval Δy (Fig. 4), the horizontal component distance x' traveled by a sound ray in that interval and the time required to travel this distance are given by.

II. 7 $$x' = \frac{\Delta y\, \bar{c}}{R_2 - R_1} \ln \left\{ \frac{\bar{c} + R_1 - (R_1 (2\bar{c} + R_1))^{1/2}}{\bar{c} + R_2 - (R_2 (2\bar{c} + R_2))^{1/2}} \right\}$$

II. 8 $$t = \frac{\Delta y}{\bar{c}\,(R_2 - R_1)} \left\{ (R_2 (2\bar{c} + R_2))^{1/2} - (R_1 (2\bar{c} + R_1))^{1/2} \right\}$$

where the quantities $\bar{c}$, R_1, R_2 and y are defined by figure 5 below.

In the case where $R_1 = R_2$ (zero gradient through the altitude increment Δy) equations II. 7 and II. 8 are indeterminate and are replaced by

II. 71 $$x' = \frac{\Delta y\, \bar{c}}{(R_1 (2\bar{c} + R_1))^{1/2}}$$

II. 81 $$t = \frac{\Delta y\, (R_1 + \bar{c})}{\bar{c}\, (R_1 (2\bar{c} + R_1))^{1/2}}$$

The additional horizontal component of sound ray travel, x'', due to the translational movement of the air mass itself is given by

II. 9 $$x'' = \bar{u}\, t$$

where, $\bar{u}$ = mean wind velocity in altitude interval Δy.

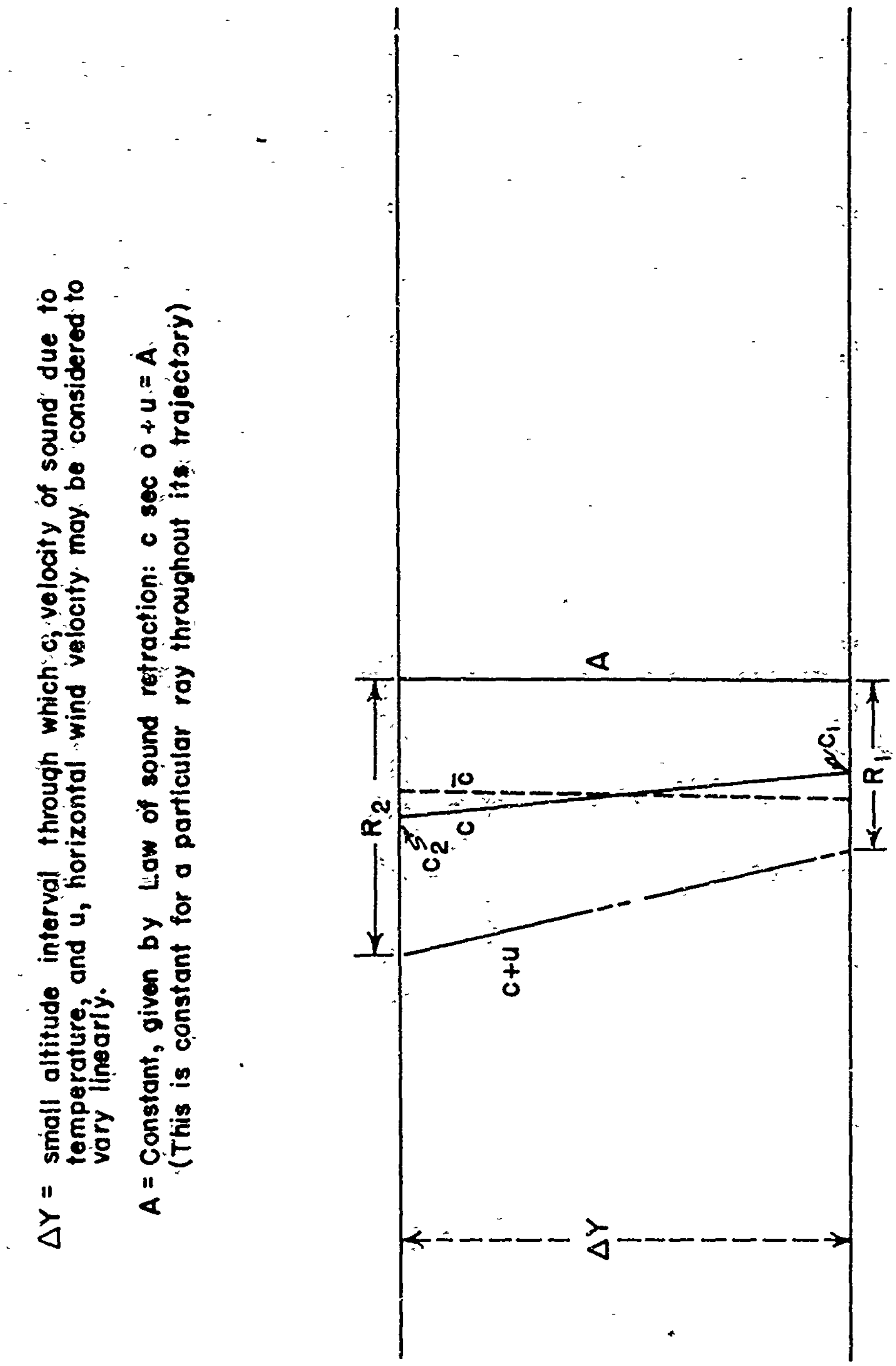

ΔY = small altitude interval through which c, velocity of sound due to temperature, and u, horizontal wind velocity may be considered to vary linearly.
A = Constant, given by Law of sound refraction: c sec θ + u = A
(This is constant for a particular ray throughout its trajectory)
A
R2
R1
c1
c̄
c
c2
c+u
ΔY

FIGURE 5

The derivations of equations II. 7 through II. 9 are given in Appendix C.

It is obvious that a summation of x', x'' and t over a series of altitude intervals (Δy's) will describe the path of a sound ray through the atmosphere.

As stated in the introduction to this paper the geographical limits of the test range for the firing of guns and explosive charges were such as to insure no blast damage to objects without the confines of this area. This statement is true even if the refraction effects of the atmosphere are considered. Thus if propagation velocity gradients in the atmosphere are such that a blast ray (sound ray) of a section of a blast wave is refracted downwards so as to strike the earth at a point distant from the source of disturbance, the normal decrease in intensity (inverse square law) with distance assures that no damaging results will occur. However, it has been noted on many occasions that damage has occured at an unreasonably great distance from the explosive source and it is this latter phenomenon which must be investigated.

Consider first the atmospheric structure shown in Figure 6a. For the sake of simplicity the propagation velocity illustrated increases linearly with increase in altitude. From the law of sound refraction it is obvious that rays of sound will be refracted downwards in such an atmosphere. Figure 6b shows the path of travel for four rays in this atmosphere. It will be noted that rays 1, 2, 3 are returned to different points along the abscissa. However, because of the change in velocity gradient above the path of ray 3, ray 4 is refracted in such a manner that it returns to the same point on the abscissa as ray 3. Now if the time of travel for rays 3 and 4 in arriving at the common point on the abscissa is the same, constructive interference will occur and an irregular pressure jump will take place. This phenomenon shall hereafter be termed "destructive focusing".

Returning now to Rothwell's equations (II. 7 to II. 9) and defining $(x' + x'')_3$, $(x' + x'')_4$, t_3 and t_4 as the horizontal distance and time of travel corresponding to the maximum altitude attained by rays 3 and 4 (Fig 6b) respectively, it is obvious that destructive focusing will occur when

II. 10
$$(x' + x'')_3 = (x' + x'')_4$$
$$t_3 = t_4$$

Because of the normal complexity of the atmospheric structure and the assumptions made in obtaining the integrated equations of motion, it would be extremely difficult to obtain theoretical solutions for equations II. 10. It is much simpler, and sufficiently valid if enough samples are taken, to obtain an approximate solution by considering various simple atmospheric structures and determining what conditions are necessary if destructive focusing is to take place.

INVESTIGATION

As pointed out in the preceeding section, the complexities normally encountered in atmospheric structures prevents a simple solution for equations II. 10. The most obvious line of investigation points to numerical examples and subsequent determination of the factors necessary to produce destructive focusing.

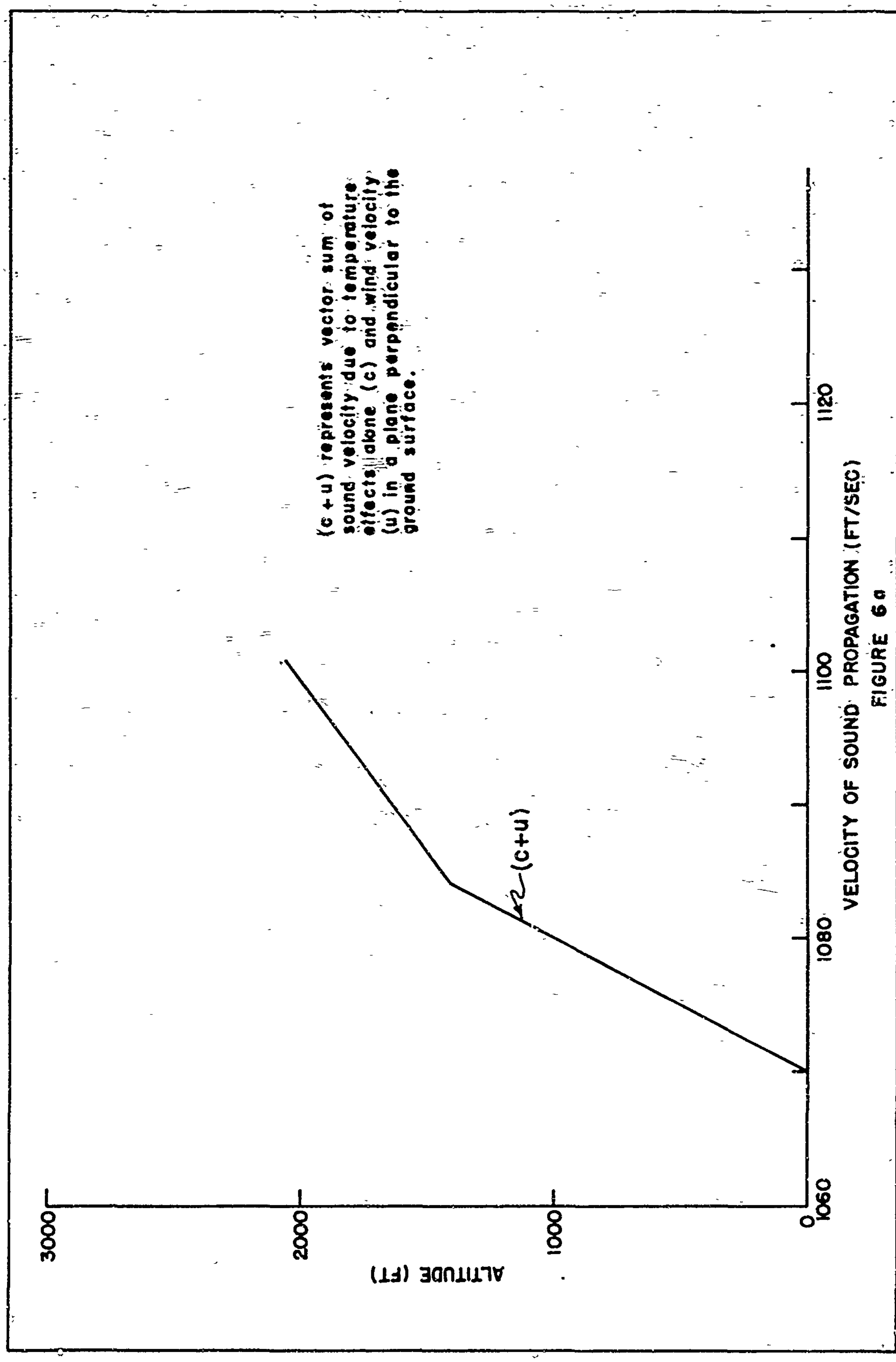

(c+u) represents vector sum of sound velocity due to temperature effects alone (c) and wind velocity (u) in a plane perpendicular to the ground surface.
(c+u)
ALTITUDE (FT)
3000
2000
1000
0
1060
1080
1100
1120
VELOCITY OF SOUND PROPAGATION (FT/SEC)

FIGURE 6a

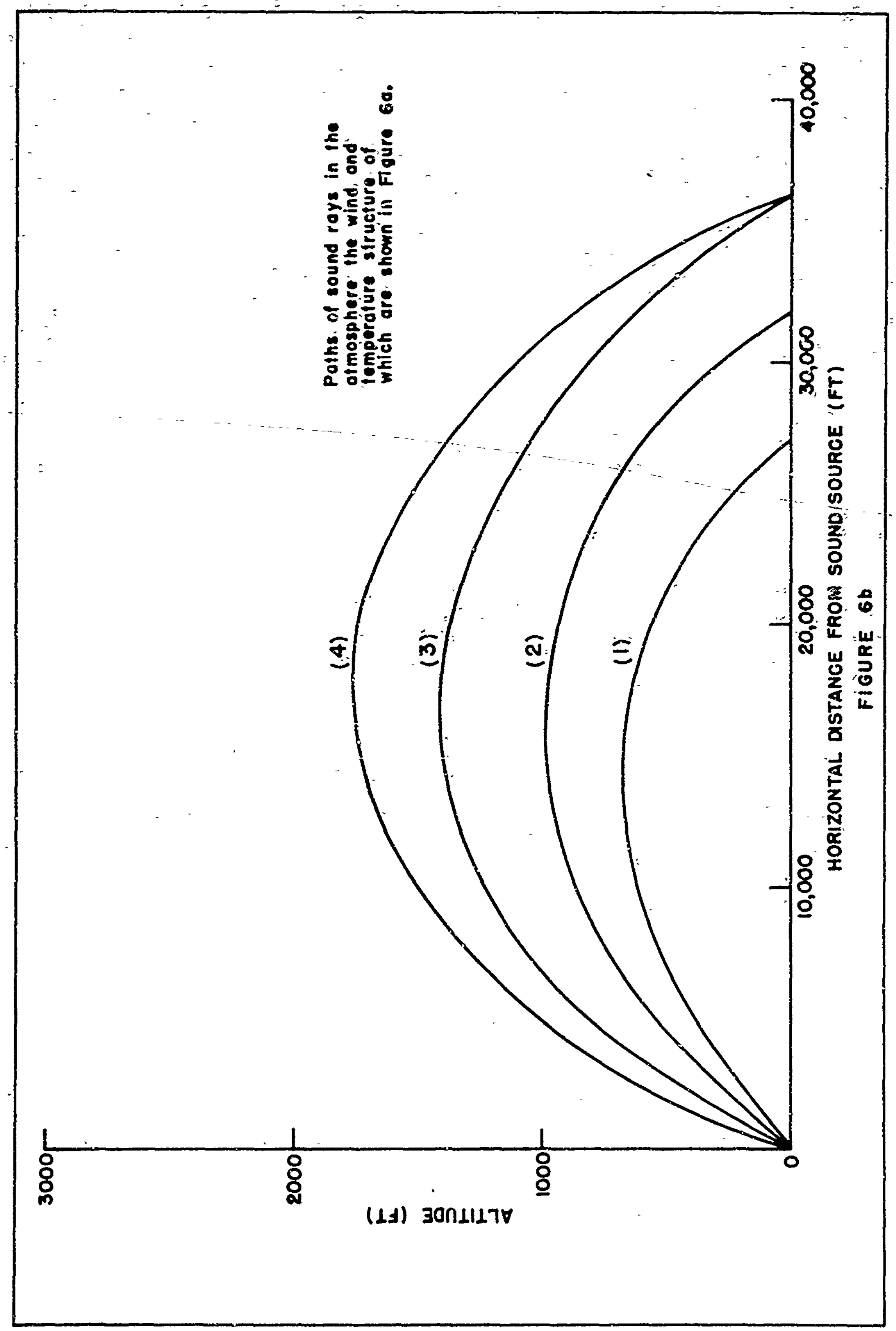

Paths of sound rays in the atmosphere the wind, and temperature structure of which are shown in Figure 6a.
(1)
(2)
(3)
(4)
ALTITUDE (FT)
3000
2000
1000
0
10,000
20,000
30,000
40,000
HORIZONTAL DISTANCE FROM SOUND SOURCE (FT)

FIGURE 6b

For the investigation three different atmospheric structures were studied (see Figure 7): positive, negative and zero gradients in temperature as a function of altitude. The horizontal distances of travel and the time of travel were calculated for several different rays of sound in each atmosphere. The Δy intervals (Figure 7) were chosen so that a particular ray of sound, ψ_1, would become horizontal (i.e. totally refracted) at an altitude equal to Δy. The atmospheric conditions in the layer above Δy necessary to focus destructively a second ray, ψ_2, with ψ_1 were then determined by Rothwell's equations and equations II. 10.

As an example consider the atmospheric structure shown in figure 7a. The atmosphere is isothermal with a temperature corresponding to a sound velocity of 1100 ft./sec. at the ground surface; the wind component along the direction of travel is -10 ft/sec. If the propagation velocity increases with altitude as is shown, it is found that the sound ray, the initial angle with the horizontal ($\theta_2°$) of which is 2°, will become horizontal at an altitude of 165 ft. The problem then is to consider the wind and temperature structure of the atmosphere which will focus destructively (with $\theta_2°$) sound rays $\theta_4°$, $\theta_6°$, $\theta_8°$ and $\theta_{10}°$ (i.e. rays, the initial angles of which were 4°, 6°, 8° and 10° respectively). Thus if x_1, t_1 and x_2, t_2 represent the horizontal distance and time of travel of rays $\theta_2°$ and $\theta_4°$ through the interval Δy, (165 ft. in this case) and if x_3, t_3 represent the same quantities for ray θ_4 in altitude interval Δy_2 above Δy_1, the conditions necessary for destructive focusing of rays $\theta_2°$ and $\theta_4°$ are

III. 1
$$x_1 = x_2 + x_3$$
$$t_1 = t_2 + t_3$$

From Rothwell's equations (II. 7 - II. 9):

III. 21
$$x_1 = x_1' + x_1'' = \left[A_1 - \frac{R_1 + R_2}{2}\right] \frac{\Delta y_1}{R_2 - R_1} \ln \left\{\frac{\bar{c}_1 + R_1 - (R_1(2\bar{c}_1 + R_1))^{1/2}}{\bar{c}_1 + R_2 - (R_2(2\bar{c}_1 + R_2))^{1/2}}\right\}$$

III. 22
$$x_2 = x_2' + x_2'' = \left[A_2 - \frac{R_3 + R_4}{2}\right] \frac{\Delta y_1}{R_4 - R_3} \ln \left\{\frac{\bar{c}_1 + R_3 - (R_3(2\bar{c}_1 + R_3))^{1/2}}{\bar{c}_1 + R_4 - (R_4(2\bar{c} + R_4))^{1/2}}\right\}$$

III. 23
$$x_3 = x_3' + x_3'' = \left[A_2 - \frac{R_5 + R_6}{2}\right] \frac{\Delta y_2}{R_6 - R_5} \ln \left\{\frac{\bar{c}_2 + R_5 - (R_5 (2\bar{c}_2 + R_5))^{1/2}}{\bar{c}_2 + R_6 - (R_6 (2\bar{c}_2 + R_6))^{1/2}}\right\}$$

III. 24
$$t_1 = \frac{\Delta y_1}{\bar{c}_1(R_2 - R_1)} \left\{(R_2 (2\bar{c}_1 + R_2))^{1/2} - (R_1(2\bar{c}_1 + R_1))^{1/2}\right\}$$

III. 25
$$t_2 = \frac{y_1}{\bar{c}_1 (R_4 - R_3)} \left\{R_4 (2\bar{c}_1 + R_4))^{1/2} - (R_3 (2\bar{c}_1 + R_3))^{1/2}\right\}$$

THREE TYPICAL ATMOSPHERIC STRUCTURES

C = VELOCITY OF SOUND PROPAGATION DUE TO TEMPERATURE EFFECTS ALONE

C+U = VECTORIAL SUM OF C AND WIND COMPONENT VELOCITY (U) ALONG LINE OF PROPAGATION CONSIDERED

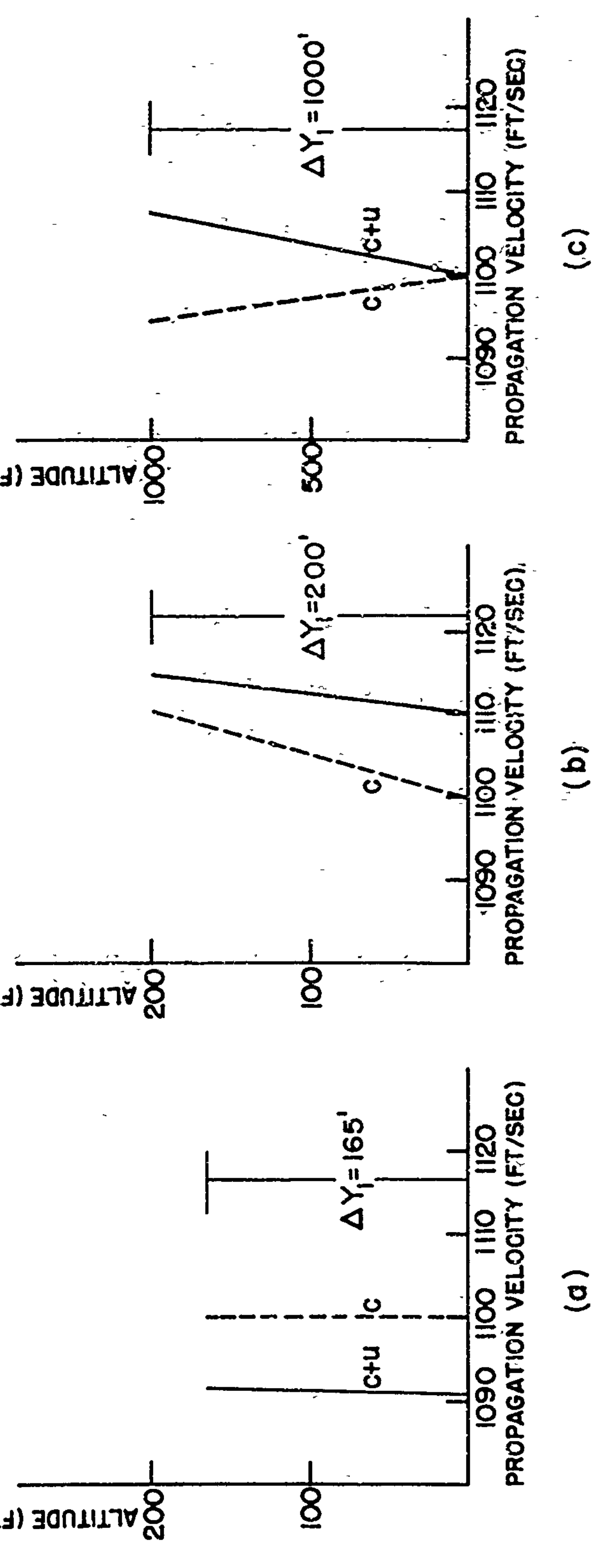


FIGURE 7

III. 26 $$t_3 = \frac{\Delta y_2}{\bar{c}_2 (R_6 - R_5)} \left\{ (R_6 (2\bar{c}_2 + R_6))^{1/2} - (R_5 (2\bar{c}_2 + R_5))^{1/2} \right\}$$

where

Law of sound refraction

$$A_1 = c_1 \operatorname{Sec} 2^\circ + u_1 = 1100\ (1.0006) - 10 = 1090.66$$

$$A_2 = c_1 \operatorname{Sec} 4^\circ + u_1 = 1100\ (1.0024) - 10 = 1092.64$$

$$y_1 = 165 \text{ ft.}$$

$$\bar{c}_1 = 1100 \text{ ft./sec.}$$

See Figure 5.

$$R_1 = A_1 - (c_1 + u_1) = 1090.66 - 1090 = 0.66$$

$$R_2 = A_1 - (c_2 + u_2) = 1090.66 - 1090\text{-}66 = 0$$

$$R_3 = A_2 - (c_1 + u_1) = 1092.64 - 1090 = 2.64$$

$$R_4 = A_2 - (c_2 + u_2) = 1092.64 - 1090.66 = 1.98$$

$$R_5 = A_2 - (c_2 + u_2) = 1092.64 - 1090.66 = 1.98 = R_4$$

$$R_6 = A_2 - (c_3 + u_3) = 1092.64 - 1092.64 = 0$$

$$y_2 = \text{unknown}$$

$$\bar{c}_2 = \text{unknown}$$

The problem as set up states that if equations III. 1 are solved, the altitude interval Δy_2, and the mean temperature $\bar{c}_2$ for that interval, may be determined for the ray $\theta_4°$, which will focus destructively with $\theta_2°$. The two quantities Δy_2 and $\bar{c}_2$ accurately define the atmospheric structure (temperature and wind gradients).

Substituting III. 2 and III. 1 gives

III. 3 $$\frac{x_1 - x_2}{\left[A_2 - \frac{R_5}{2}\right] \ln \left\{ \frac{\bar{c}_2 + R_5 - (R_5 (2\bar{c}_2 + R_5))^{1/2}}{\bar{c}_2} \right\}} = \frac{(t_1 - t_2)\,\bar{c}_2}{-\left\{ R_5 (2\bar{c} + R_5) \right\}^{1/2}}$$

x_1, x_2, t_1 and t_2 are easily solved from the numerical values given after equations III. 2. Since R_5 is also known, equation III. 3 may be solved to find the value of $\bar{c}_2$. Once this is done substitution of $\bar{c}_2$ into either equation III. 23 or equation III. 26 will result in a solution for Δy_2 and the atmospheric structure will be determined. In the numerical example above Δy_2 is found to be 209.0 ft. and $\bar{c}_2$ equal to 1100. 15 ft./sec.

The solution for $\bar{c}_2$ in equation III. 3 may be accomplished, without too much difficulty, by graphical methods. However, another solution may be found (see Appendix D) for the quantities Δy_2 and $\bar{c}_2$ by the following formulae:

III. 41 $$\bar{c}_2 = \frac{R_5}{\cos h \frac{m}{\Delta y_2} - 1} \qquad m = \frac{2 (R_5)(x_1 - x_2)}{2 A_2 - R_5}$$

III. 42 $$\Delta y_2 = m\sqrt{6 \left(\frac{m}{n} - 1\right)} \qquad n = R_5 (t_1 - t_2)$$

The equations III. 41 and III. 42, while exact, necessitate a determination of t_1 and t_2 accurate to five significant figures. Unfortunately the simplified equations of Rothwell's are not always accurate to this number of significant figures so that results obtained in solving for $\bar{c}_2$ and Δy_2 by means of equations III. 41 and III. 42 may oftentimes be totally invalid. For this reason the graphical solution for equation III. 3 was adhered to in all calculations.

Details of the calculations involved in studying the atmospheric structures of Figure 7 were not considered essential for the main body of the report and consequently these may be found at the end of the report in Appendix E.

CONCLUSIONS

The last section dealt very briefly with the method used in investigating those atmospheric conditions which are conducive to destructive focusing. The description was purposely kept brief because of the relative unimportance of the mathematical gymnastics. The numerical results for one computation were considered sufficient to illustrate the method.

The systematic investigations of the three ideal atmospheres pictured in Figure 7 revealed one very significant and consistent result. In order that destructive focusing of two or more sound rays might occur, a temperature inversion in the atmosphere was found necessary. Referring to Figures 7a and 7c, a positive temperature gradient above the altitude increment Δy_1 was a necessary condition for destructive focusing to take place. For the atmosphere illustrated in Figure 7b, a negative or positive (or zero) temperature gradient above the altitude interval Δy_1 was a necessary condition. Actually the results from the study of Figure 7b might sound ambiguous, but by definition a temperature inversion in the atmosphere indicates merely the existence of a positive temperature gradient. If one particular line of propagation (direction from the source) is considered — such as the examples in Figure 7 — certain conditions must also be fulfilled in the wind gradient structure. However, the problem is concerned with destructive blast propagation in any

direction and consequently the wind component, u_1 may vary infinitely for each atmospheric structure existent in nature. Thus it is only practicable to determine the necessary rather than the sufficient condition.

Earlier in the paper, mention was made of the fact that water vapor present in the atmosphere also has an effect on sound propagation. Since this element (water vapor) varies only with altitude — neglecting local and low level influences, such as a body of water — its effect may be considered simultaneously with the temperature gradient. In Appendix A, formulae are derived for the propagation of sound in dry and moist air. Figure 8 below illustrates graphically the effect of water vapor on the velocity of sound at sea level and and at 3000 feet of altitude. Thus in constructing the curve showing the variation of sound velocity with altitude as a function of temperature alone it is necessary to consider the correction due to the presence of water vapor before the true picture can be given of variation of propagation velocity with altitude. As can be seen from figure 8 the mere presence of water vapor in the atmosphere would not seriously affect the temperature-propagation curve, but a vertical gradient in the relative humidity might possibly change the temperature-propagation curve from one of a negative or zero gradient to a positive gradient and thus bring about a sufficient condition for destructive focusing.

Although the existence of a positive temperature gradient may be absolutely determined only by an actual sounding of the atmosphere, there are many meteorological phenomena, observable from the ground, which indicate its presence. The mere fact that the normal or average temperature structure in the atmosphere is characterized by a negative gradient strongly suggests the association of abnormal gradients with observable abnormalties. A discussion of pertinent meteorological elements is given below.

1. **Wind effects.** A strong horizontal wind tends to produce considerable eddy currents either near large physical objects on ground surfaces or along shear surfaces in the atmosphere itself. These eddy effects thoroughly mix the atmospheric elements and produce the normal negative temperature gradient. The same eddy effects produce vertical wind currents and would distort sound wave fronts in an unpredictable manner. Since the entire theory of refraction in the atmosphere of sound or blast waves is based upon the assumption of laminar flow with no vertical currents, the eddy effects produced by the strong horizontal flow of air would nullify any results of the above investigation. However, the distortion of sound waves produced by such a flow would make destructive focusing extremely improbable and the presence of strong winds reasonably indicates conditions not conducive to destructive focusing. Strong surface winds aloft, and, although conclusive evidence is not at the moment available, surface winds of 15 mph or greater seem likely meteorologically safe firing conditions.

2. **Temperature Inversions** A temperature inversion in the atmosphere accompanies numerous synoptic weather phenomena but most of these phenomena are associated with a rather special synoptic pattern. In the preceding paragraph the assertion was made that a strong horizontal air flow tended to produce a negative temperature gradient in the atmosphere. Consequently, if temperature inversions are to exist the horizontal transport of air must be small. In general, light winds — up to any appreciable altitude — are found within high pressure areas which tend to stagnate for periods of several days' duration over a large ground area. If the high pressure cell on the earth's surface is small, say of the order of 100 miles in diameter, and is moving fairly rapidly, the winds although light or calm at the surface will be found to be high enough at 2000-3000 feet in altitude to produce mixing and consequently a negative temperature gradient at that altitude. True, close to the earth's surface, there may be a temperature

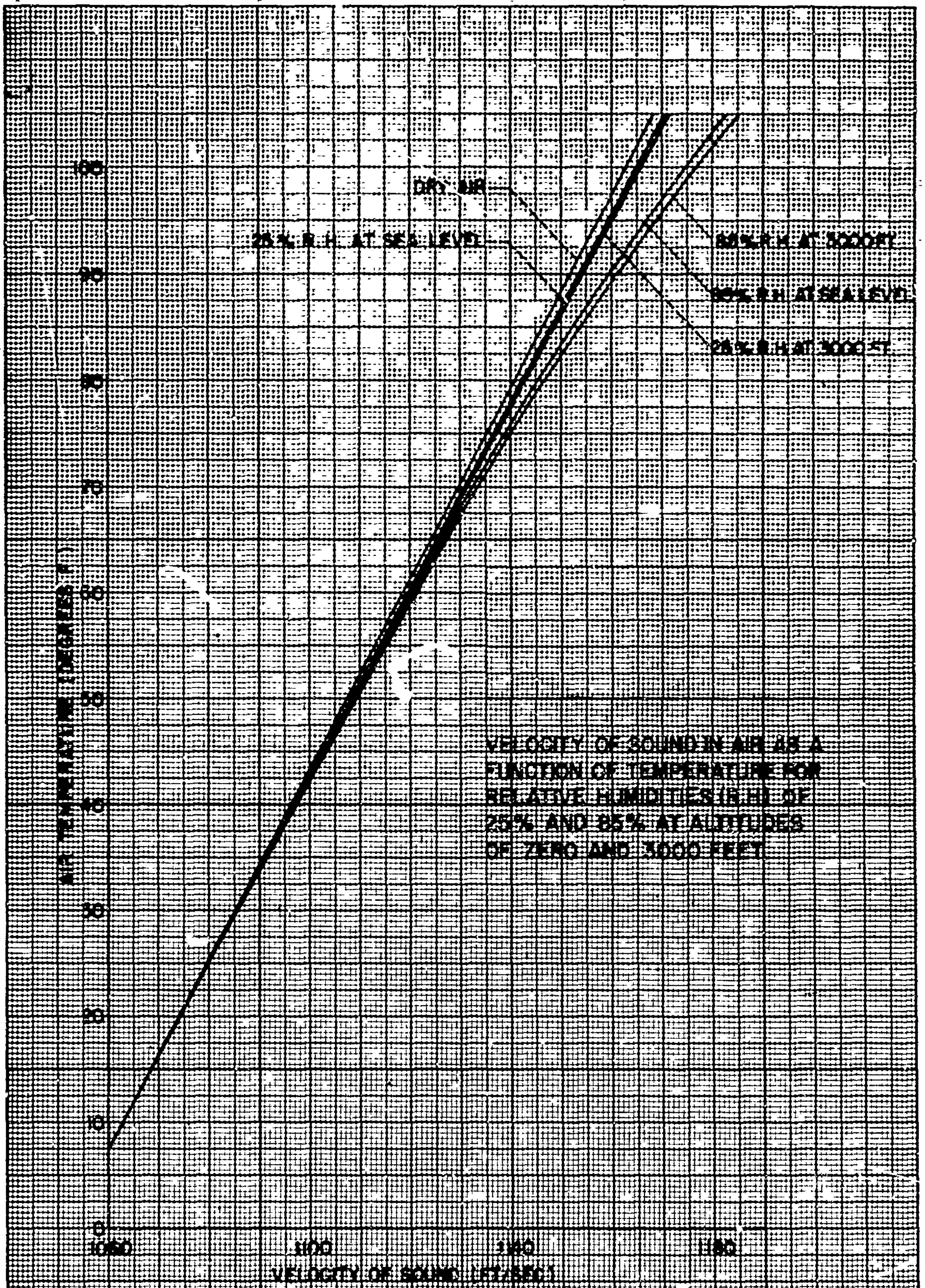

DRY AIR
25% R.H. AT SEA LEVEL
85% R.H. AT 3000 FT.
85% R.H. AT SEA LEVEL
25% R.H. AT 3000 FT.
VELOCITY OF SOUND IN AIR AS A FUNCTION OF TEMPERATURE FOR RELATIVE HUMIDITIES (R.H.) OF 25% AND 85% AT ALTITUDES OF ZERO AND 3000 FEET
AIR TEMPERATURE (DEGREES F)
0
10
20
30
40
50
60
70
80
90
100
1060
1100
1140
1180
VELOCITY OF SOUND (FT/SEC)

FIGURE 9

inversion due to the greater radiation of the ground (compared to the air) but this inversion would be sufficiently low to keep destructive effects within the blast range area. The large scale high pressure area which stagnates for several days over portions of the earth's surface is characterized by the subsiding or sinking of the air mass and a corresponding increase in the temperature gradient (less negative if originally negative and more positive if originally positive) caused by the adiabatic heating in compression. The wind velocities are fairly low up to altitudes of perhaps 6000 or 7000 feet and the atmosphere becomes quite stratified. This condition is quite conducive to destructive focusing and there is little distortion of sound wave fronts in passing through the atmosphere. These extensive areas of high pressure tend to be more frequent and more persistent in the summer and winter months and they may easily be recognized either by reference to the current weather map or by observing meteorological elements such as temperatures, visibilities and cloud types.

There exist weather situations in which temperature inversions occur in low pressure areas in the presence of fairly high wind velocities. The presence of warm or cold frontal surfaces in the immediate vicinity indicate temperature inversions at the boundry between the different air masses. However, the slope of the frontal surface implies a horizontal temperature gradient as well as a vertical one and the laws of sound refraction again do not apply. Of course the immediate presence of a warm or cold front usually indicates rather adverse weather conditions — rain, snow, high winds, poor visibility, etc. — and the relatively rapid movement of the frontal surface, in a horizontal direction, would make destructive focusing rather difficult in view of the wind discontinuity at the frontal surface.

Ultimately the decision as to whether or not a firing is to take place should be based upon a sounding of the atmosphere by suitable meteorological instruments. However, a perusal of the preceeding paragraphs enables one to make some good guesses as to the possibility of destructive focusing for a given atmospheric situation. Some rules of thumb might be expressed as follows:

1. A high pressure area which is stationary over the area in question becomes dangerous after the first day of its presence.
2. Large diurnal temperature variations at the earth's surface indicate unfavorable firing conditions.
3. Poor visibilities and light winds in the early morning followed by cloudless conditions throughout the day indicate dangerous focusing possibilities.
4. Poor visibilities and light winds in the early morning followed by the formation of cumulus (swelling) clouds in the late morning and early afternoon indicate safe conditions for an afternoon firing.
5. Light surface winds and low relative humidities at the ground plus the presence of a stratified cloud deck below 10,000 feet indicate dangerous conditions for firing.

When soundings of the atmosphere are to be taken a rather important question arises as to how high or to what altitude the atmosphere must be examined. This question cannot be answered simply because of the dependence of the limits of destruction upon the weight of the detonating charge (See Sachs' I. 1). However, by means of equation II. 7 one may plot x' as a function of Δy and $(R_2 - R_1)$ and assuming various values for the quantity $\bar{c}$. It is foun that over a normal range of values, variations in $\bar{c}$ have little effect on the dependent variable x' and may be considered constant for our purposes. Now if R_2 is set equal to 0, x' will represent the horizontal distance traversed by a sound ray from the origin (source of disturbance) to the point where the normal to the wave front becomes horizontal and thus $2x'$ would define the point at

which the sound ray returns to the earth's surface (neglecting earth curvature effects). Figure 9 below represents the plot of equation II. 7. The physical significance of such a graph is that if the actual velocity gradient of sound existent in the atmosphere is superimposed upon the graph, only that portion of the gradient curve which lies between the abscissa and the parametric curve $2x' = n$ need be considered in determining whether or not destructive focusing will occur between the origin and and $2x' = n$. For example, supposing the propagation velocity gradient along a certain direction from a detonating source is represented by curve in Figure 9. For a particular weight of explosive we find by Sach's formula, I. 1, that the destructive effects will be confined to a circular area of, say, 10 mile radius (i.e. $2x' = 10$ miles). In Figure 9, the gradient curve crosses the parametric curve $2x' = 10$ at an altitude of 3000 feet. Therefore, only the region of the atmosphere below 3000 feet need be considered for determining the existence of destructive focusing. By means of this plot and the meteorological data on vertical velocity gradients existing 12 hours or less before the planned firing time (available from local weather stations), a rough estimate of the height to which the atmosphere must be sounded (to determine the existence of destructive focusing) is obtained.

The preceeding paragraphs have been concerned mainly with implications rather than objective facts pertaining to the problem of destructive focusing. It is felt that such a superficial discussion is permissible within the scope intended for such a preliminary investigation. However, an attempt was made to prove some of the conclusions stated in earlier paragraphs by examining meteorological conditions on days when fairly valid complaints were received from inhabitants living outside the Proving Ground area. Three examples of typical complaints were chosen at random (February 20, 1944, March 16, 1944 and March 26, 1945) and the atmospheric conditions examined. In each case the following conditions were noted:

1. A positive temperature gradient existed at some point in the atmosphere (at relatively low altitudes).
2. Surface winds were less than 8 mph.
3. Visibilities were poor in either fog or smoke.
4. A high pressure area (surface) characterized the synoptic situation.

This evidence, while far from conclusive, does certainly point to atmospheric conditions conducive to destructive focusing.

On January 14, 1948, a very clear cut case of destructive focusing was encountered and accurately reported. In this instance a 12,000 lb. charge was detonated in the face of a surface wind blowing from the north at approximately 14 mph. At 4500 yards to the north only a faint sound was heard; 15,000 yards to the north no sound was heard; 25,000 yards to the north a sharp crack followed by a prolonged rumbling was noted by numerous observers. The normal decrease in intensity of the blast wave moving up wind was quite in line with the observations at 4500 and 15,000 yards but some change in atmospheric structure aloft was necessary to produce the observed effects at 25,000 yards. Figure 10 illustrates the sound propagation gradient along the line from the detonating source to the point at 25,000 yards where the report was heard. From the figure it is easily seen that the necessary condition for destructive focusing existed.

One final comment should be added at this point. It has been suggested that the destructive effects of large detonating charges might be attributed to shock transmitted by the ground. There are two facts which strongly discredit this possibility:

1. If the transmitted shock were a surface wave (in the earth's crust) it would decrease approxi-

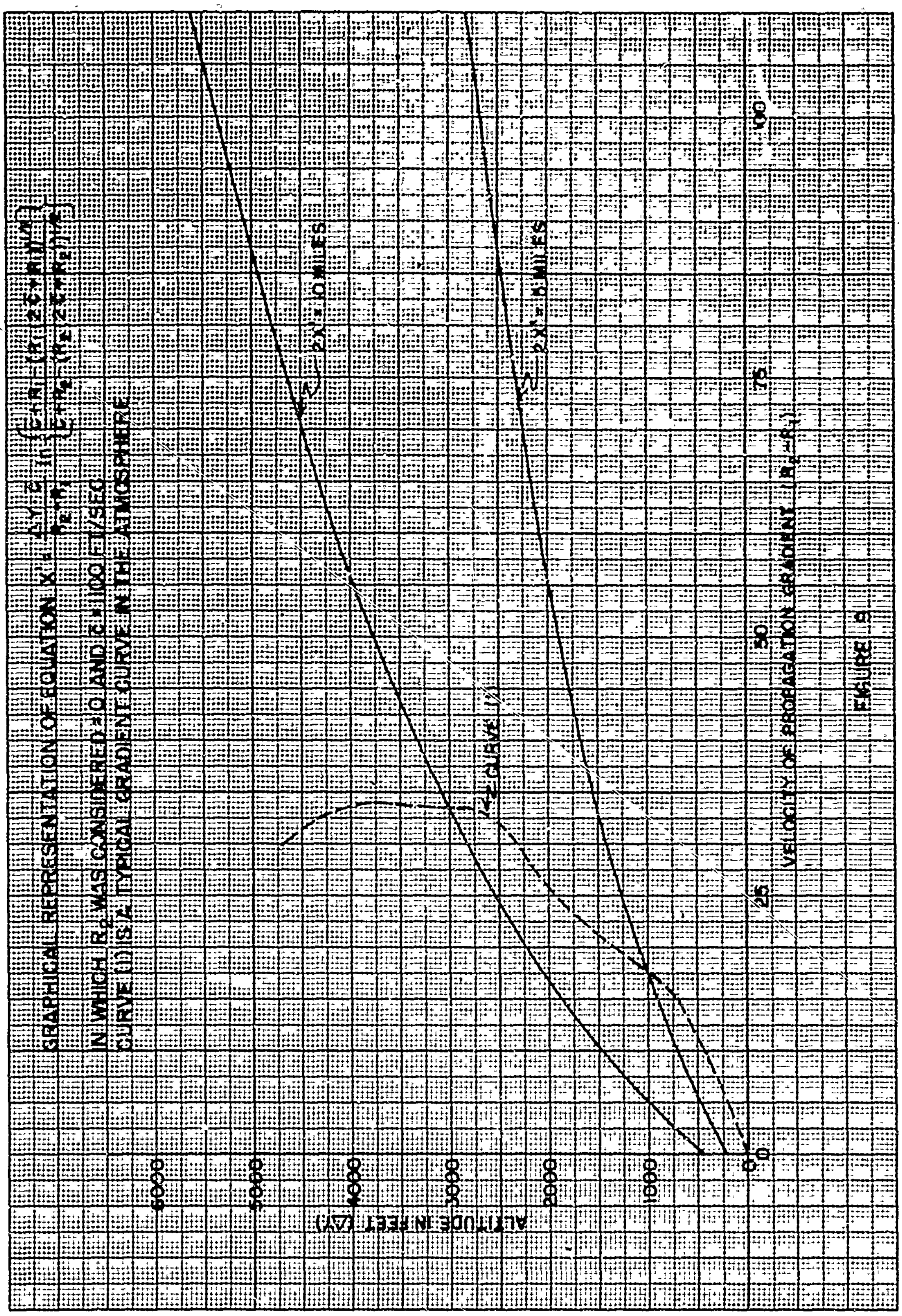

GRAPHICAL REPRESENTATION OF EQUATION $X' = \frac{\Delta Y \cdot C}{R_2 - R_1} \ln \left\{ \frac{C + R_1 - (R_1(2C + R_1))^{1/2}}{C + R_2 - (R_2(2C + R_2))^{1/2}} \right\}$
IN WHICH R_2 WAS CONSIDERED = 0 AND C = 1100 FT/SEC
CURVE (1) IS A TYPICAL GRADIENT CURVE IN THE ATMOSPHERE
ALTITUDE IN FEET (ΔY)
6000
5000
4000
3000
2000
1000
0
0
25
50
75
100
VELOCITY OF PROPAGATION GRADIENT ($R_2 - R_1$)
2X' = 10 MILES
2X' = 5 MILES
CURVE (1)

FIGURE 9

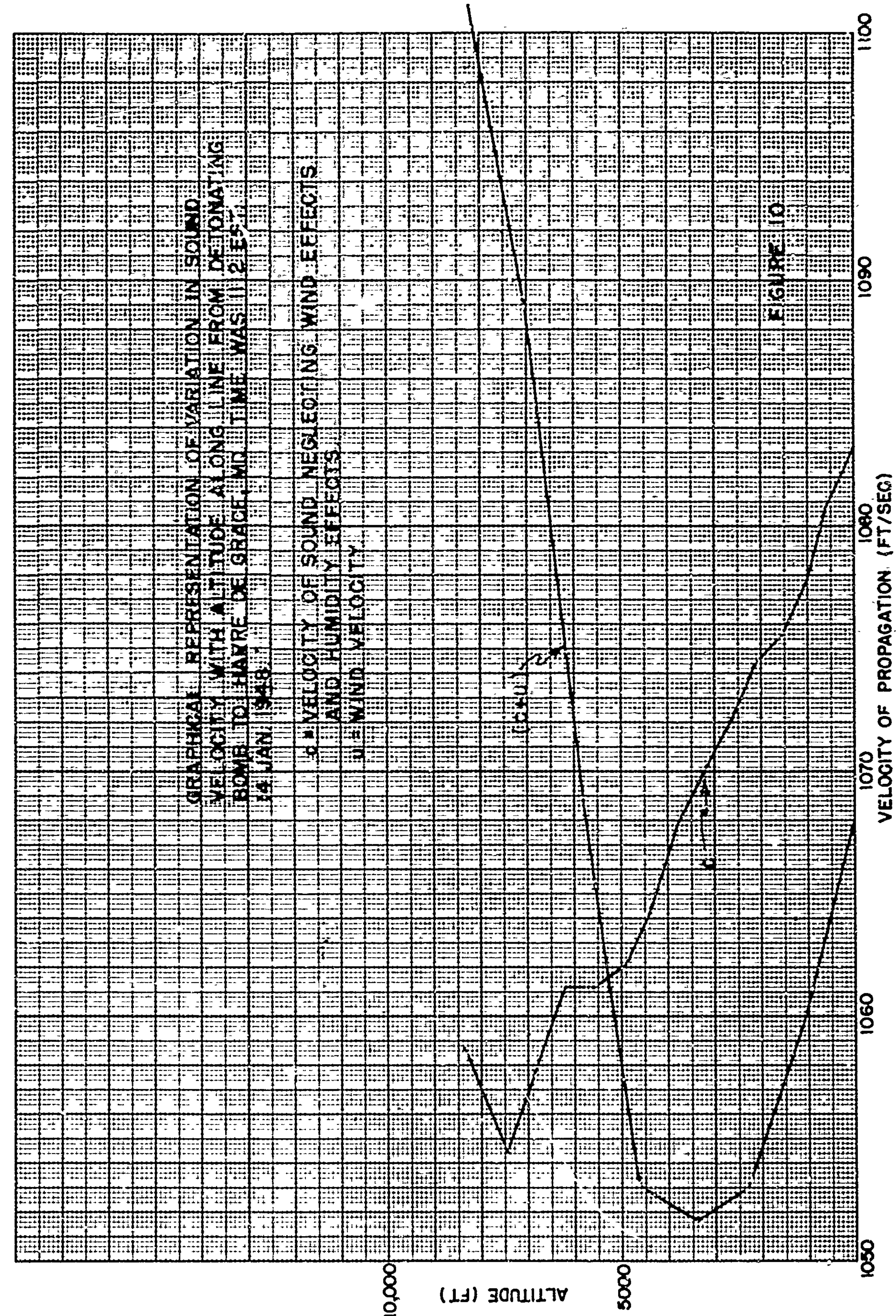

GRAPHICAL REPRESENTATION OF VARIATION IN SOUND VELOCITY WITH ALTITUDE ALONG LINE FROM DETONATING BOMB TO HAVRE DE GRACE, MD. TIME WAS 1112 EST. 14 JAN. 1948.
c = VELOCITY OF SOUND NEGLECTING WIND EFFECTS AND HUMIDITY EFFECTS.
u = WIND VELOCITY
(c+u)
c
1050
1060
1070
1080
1090
1100
VELOCITY OF PROPAGATION (FT/SEC)
ALTITUDE (FT)
5000
10,000

FIGURE 10

mately as the inverse square distance from the source and residents living close to the Proving Ground would suffer far in excess of those living at some distance from the area — a fact not observed in actual cases.

2. If the shock wave were reflected or refracted from some interface or change in structure below the earth's surface, a particular location would suffer equally from every charge (of equal magnitude) detonated at the same point on the Proving Ground — likewise a fact not observed during firings.

Warren W Berning

W. Berning

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APPENDIX A

Derivation of the formulae for the propagation of sound in the atmosphere (References (14) and (15))

Consider an imaginary tube of unit cross section parallel to the line of propagation and containing a portion of the sound wave front. Let F be the external force supplied so as to keep the form of the wave front unchanged and moving with a uniform velocity a. The momentum entering one end of the imaginary tube per interval of time is equal to the momentum leaving the tube during the same time interval. Thus if P represents the external atmospheric pressure, ΔP the pressure increase in the sound wave, a, the velocity of entry of air of density ρ and $a \mp u$ and ρ_1 the velocity and density of the air as it leaves the tube

(A. 1) $$M(a \mp u) + P \mp \Delta P + F = Ma + P$$

or

(A. 2) $$\rho a u = P \pm F.$$

Now, if F is equal to 0, the case which corresponds very closely to sound propagation in the atmosphere since the pressure behind the sound wave is almost exactly equal to that ahead of it

(A. 3) $$\rho a = \frac{\Delta P}{u}$$

or

(A. 4) $$a = \frac{\Delta P}{\rho u}$$

However, from the statement of the problem

(A. 5) $$\rho_1 (a \mp u) = \rho a$$

and if the compressional changes take place adiabatically ρ_1 is given by

(A. 6) $$\frac{P + \Delta P}{P} = \left(\frac{\rho_1}{\rho}\right)^{\gamma}, \quad \gamma = \text{ratio of specific heats in air}$$

so that equation (A. 4) may be written

(A. 7) $$a = \sqrt{\frac{\gamma P}{\rho}}$$

if we neglect the product $\Delta P \cdot u$.

Equation (A. 7) is the general expression for the velocity of sound in a dry atmosphere where the density ρ of the atmosphere is dependent only upon temperature and pressure.

In practically treating the problem of sound propagation through the atmosphere, account must be taken of the presence of water vapor. Because of its composition water vapor has a lower density than the air which it replaces and consequently moist air is lighter than dry air. The equation of state for a perfect gas is given by

(A. 8) or,

$$P = \frac{R^*}{M} \rho T$$

$$\rho = \frac{MP}{R^* T}$$

T = absolute temp.
ρ = density
P = pressure
M = molecular wt. of gas
R^* = universal gas const.

Dalton's Law states that the sum of the partial pressures of a mixture of gases is equal to the total pressure. Thus if ρ_m is the density of a mixture of gases of densities ρ_d (dry air) and ρ_w (water vapor), the equation of state (A. 8) becomes

(A. 9)

$$\rho_m = \frac{M(P-\epsilon)}{R^* T} + \frac{M w(\epsilon)}{R^* T}$$

M w = molecular wt. of water vapor

$$= \frac{PM}{R^* T}\left(1 - \frac{0.379\epsilon}{P}\right)$$

ϵ = partial pressure of water vapor

Substitution of ρ_m into equation A. 7 and designating a and a' as the velocity of sound in dry and moist air respectively, one obtains

(A. 10)

$$a' = a \sqrt{\frac{1}{1 - 0.379\,\epsilon/P}}$$

If now, the medium in which the sound wave is being propagated is itself moving relative to a coordinate reference system (such as the earth) the absolute velocity (with respect to the earth) of the sound wave, designated as a'' is given simply by

(A. 11)

$$a'' = a' \pm \bar{R} = a \sqrt{\frac{1}{1 - 0.379\,\epsilon/P}} \pm \bar{R}$$

$\bar{R}$ = wind velocity normal to wave front.

APPENDIX B

Condensed derivation of the "Law of Sound Refraction" as taken from Milne's work (Ref. (3))

Let: u, v, w be the components of velocity at point (x, y, z) in the atmosphere moving with a continuous flow; a be the velocity of sound, a function of x, y, z; l, m, n be the direction cosines of the normal to the wave front at point (x, y, z). Then the equations of a sound ray are given by

B. 1 $$\frac{dx}{dt} = la + u, \quad \frac{dy}{dt} = m\,a + v, \quad \frac{dz}{dt} = n\,a + w$$

where t is the time.

The ray, however, is not simply obtained by a simple integration of equations B. 1 since l, m and n are known only at their initial values. Therefore let (x, y, z) and (x + δ x, y + δ y, z + δ z) represent two points of intersection of neighboring sound rays with one sound wave front at time t. Subtracting pairs of equations of type B. 1 one obtains,

$$\frac{d(\delta x)}{dt} = \delta\,(la + u) = \sum \delta x \frac{\delta}{\delta x}(la + u)$$

B. 2 $$\frac{d(\delta y)}{dt} = \delta\,(ma + v) = \sum \delta y \frac{\delta}{\delta y}(ma + v)$$

$$\frac{d(\delta x)}{dt} = \delta\,(na + w) = \sum \delta x \frac{\delta}{\delta z}(na + w)$$

Now differentiating the relation

B. 3 $$l\,\delta x + m\,\delta y + n\,\delta z = 0$$

with respect to time and substituting from B. 2 for d (δ x)/dt, etc., one obtains, after reduction,

B. 4 $$\sum \delta x \left(\frac{dl}{dt} + \frac{\partial a}{\partial x} + l\,\frac{\partial u}{\partial x} + m\,\frac{\partial v}{\partial x} + n\,\frac{\partial w}{\partial x} \right)$$

Equation B. 4 holds for all values of δx; δy; δz satisfying B. 3, hence,

B. 5

$$\frac{1}{l}\left(\frac{dl}{dt} + \frac{\partial a}{\partial x} + l\,\frac{\partial u}{\partial x} + m\,\frac{\partial v}{\partial x} + n\,\frac{\partial w}{\partial x}\right)$$

$$= \frac{1}{m}\left(\frac{dm}{dt} + \frac{\partial a}{\partial y} + l\,\frac{\partial u}{\partial y} + m\,\frac{\partial v}{\partial y} + n\,\frac{\partial w}{\partial y}\right)$$

$$= \frac{1}{n}\left(\frac{dn}{dt} + \frac{\partial a}{\partial z} + l\,\frac{\partial u}{\partial z} + m\,\frac{\partial v}{\partial z} + n\,\frac{\partial w}{\partial z}\right)$$

Equations B. 1 and B. 5 give the ray completely. Equation B. 5 may be expressed more simply as follows. Let

$$V = a + l\,u + m\,v + n\,w$$

and let l, m and n be considered as constant in differentiating with regard to x, y, z. Then B. 5 may be written

B. 6

$$\frac{1}{l}\left(\frac{dl}{dt} + \frac{\partial V}{\partial x}\right) = \frac{1}{m}\left(\frac{dm}{dt} + \frac{\partial V}{\partial y}\right) = \frac{1}{n}\left(\frac{dn}{dt} + \frac{\partial V}{\partial z}\right)$$

$$= V\left(l\,\frac{\partial}{\partial x} + m\,\frac{\partial}{\partial y} + n\,\frac{\partial}{\partial z}\right),$$

or explicitly,

B. 7

$$\frac{dl}{dt} = -\left(m^2 + n^2\right)\frac{\partial V}{\partial x} + l\,m\,\frac{\partial V}{\partial y} + l\,n\,\frac{\partial V}{\partial z})$$

Equations B. 1 express the direct propagation of the sound, as influenced by the bodily displacement of the medium, while equations B. 6 express the refraction of the sound caused by the variation of the normal speed V over the surface of the wave front.

If, as in this paper, we are concerned only with the motion taking place within a stratified medium, the z-axiscan always be chosen so that the velocity component $w = 0$ and u, v, a are functions of z only. In this case equations B. 1 and B. 6 reduce to

B. 8 $$\frac{dx}{dt} = la + u, \quad \frac{dy}{dt} = m a + v, \quad \frac{dz}{dt} = n a$$

B. 9 $$\frac{1}{l}\frac{dl}{dt} = \frac{1}{m}\frac{dm}{dt} = \frac{1}{n}\left(\frac{dn}{dt} + \frac{\partial V}{\partial z}\right) = n\frac{\partial V}{\partial z}$$

$$V = a + l\,u + m\,v$$

The first of equations B. 9 integrates immediately in the form

B. 10 $$\frac{l}{m} = \text{constant.}$$

Equation B. 10 states that in a stratified medium the normals to the wave fronts along a particular sound ray remain parallel to a fixed plane which is itself perpendicular to the planes of stratification (in this instance the planes parallel to the xy plane).

If now one chooses the x, y axes so that the projections of the normals are parallel to the x axis, $m = 0$ and so

$$\frac{dz}{dt} = n\,a, \quad \frac{dl}{dt} = l\,n\left(\frac{\partial a}{\partial z} + l\,\frac{\partial u}{\partial z}\right),$$

whence,

$$\frac{dl}{dz} = \frac{l}{a}\left(\frac{\partial a}{\partial z} + l\,\frac{\partial u}{\partial z}\right)$$

or,

B. 11 $$-\frac{d}{dz}\left(\frac{a}{l}\right) = \frac{du}{dz}$$

Equation B. 11 integrates in the form

$$\frac{a}{l} - \frac{a_o}{l_o} = u_o - u$$

or, putting $l = \cos\theta$, $l_o = \cos\theta_o$,

B. 12 $$a \sec\theta + u = a_o \sec\theta + u_o = \text{const.}$$

Equation B. 12 is the familiar expression for the law of sound refraction.

APPENDIX C

Derivation of Rothwell's equations (see reference (13)) for the motion of a sound ray in the atmosphere.

The "refraction law" states that

C. 1 $$c \sec \theta + u = A,$$

c = velocity of sound in calm air
u = horizontal wind velocity component
A = constant
θ = angle between horizontal and normal to the wave front.

This equation may be written in the form

C. 2 $$\sec \theta = (R/c) + 1,$$

where

C. 3 $$R = A - (c + u).$$

Figure C—1 represents a "linearized" vertical segment of the atmosphere (explained in the text). In Figure C—1 let u, c and R at the layer boundaries be distinguished by subscripts 1 and 2, and let

C. 4 $$c_1 - c = hy$$

C. 5 $$u_1 - u = ky$$

y being measured from boundary 1. Δy is the thickness of the layer.

From equations C. 1, C. 4 and C. 5 are derived the following:

C. 6 $$y = \frac{c_1 \sec \theta - (A - u_1)}{h \sec \theta + k}$$

C. 7 $$\frac{dy}{d\theta} = \frac{\left\{c_1 k + (A - u_1) h\right\} \sin \theta}{(h + k \cos \theta)^2}$$

C. 8 $$u = \frac{A k + (u_1 h - c_1 k) \sec \theta}{h \sec \theta + k}$$

C. 9 $$C = \frac{c_1 k + (A - u_1) h}{h \sec \theta + k}$$

In Figure C-2 which represents a ray PQ' in an elementary layer of infinitesimal thickness δy, PQ is the direction of the wave normal and QQ' is the component wind displacement of an element of wave front while traversing the layer. The velocity in the layer is c, hence the time δt taken by the element to cross the layer is

$$\delta t = \delta y / c \sin \theta .$$

also

$$\delta x = \delta x' - \delta x'' = \delta y \cot \theta + u \, \delta t.$$

REPRESENTATION OF C, u, AND R AT THE BOUNDARIES OF A FINITE LAYER OF THICKNESS ΔY

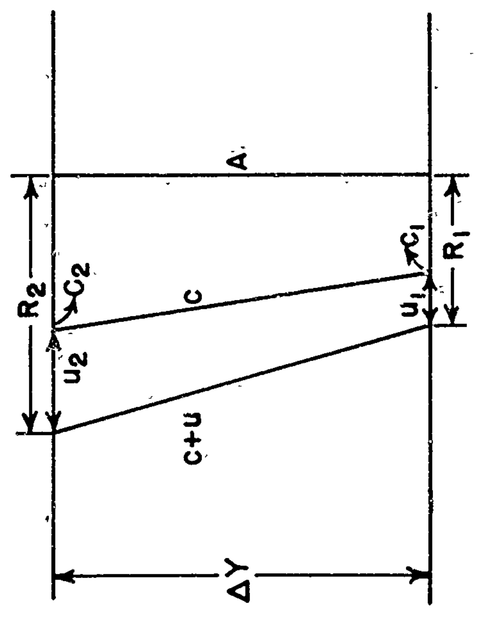

FIGURE C-1

RAY DIAGRAM IN AN ELEMENTARY LAYER

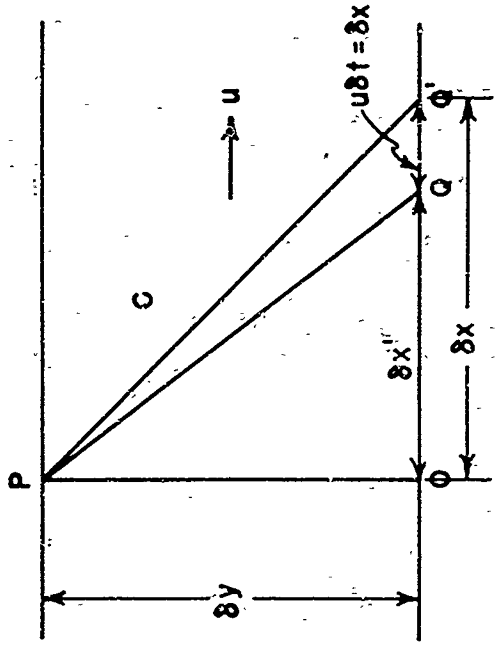

FIGURE C-2

In the limit

$$\frac{dx'}{d\theta} = \frac{dy}{d\theta}\cot\theta$$

or from equation C. 7

C. 10 $$\frac{dx'}{d\theta} = \frac{\left\{c_1 k + (A - u_1) h\right\}\cos\theta}{(h + k\cos\theta)^2}$$

and

$$\frac{dx''}{d\theta} = u\frac{dt}{d\theta} = u\frac{dy}{d\theta}\left(\frac{1}{c\sin\theta}\right)$$

or from equations C. 7, C. 8 and C. 9

C. 11 $$\frac{dx''}{d\theta} = \frac{u_1 h - c_1 k + A k\cos\theta}{\cos\theta\,(h + k\cos\theta)^2},$$

also

$$\frac{dt}{d\theta} = \frac{dy}{d\theta}\left(\frac{1}{c\sin\theta}\right)$$

or from equations C. 7 and C. 9

C. 12 $$\frac{dt}{d\theta} = \frac{1}{\cos\theta\,(h + k\cos\theta)}$$

On integrating between the boundaries of the finite layer, equation C. 10 becomes

C. 13 $$x' = \frac{2\left\{c_1 k + (A - u_1) h\right\}}{h^2 - k^2}\left\{\left(\frac{h}{h-k}\right)\left[\tan\theta/2/\tan^2\frac{\theta}{2} + \frac{h+k}{h-k}\right]_{\theta_1}^{\theta_2} - \frac{k}{(h^2 - k^2)^{1/2}} \times \left[\tan^{-1}\left(\frac{h-k}{h+k}\right)\tan\frac{\theta}{2}\right]_{\theta_1}^{\theta_2}\right\},$$

if $h^2 > k^2$.

When $h^2 < k^2$ the form of the solution is different and depends on whether $(h + k)/(h - k) >$ or $<$ $\tan^2\frac{\theta}{2}$. However, the solution given above is sufficient for the argument. Substitution into equation C. 13 from C. 2, C. 3, C. 4 and C. 5 gives the result,

$$x' = \frac{\Delta y}{(R_2 - R_1)(2c_1 + R_1 - 2c_2 - R_2)}\left[\left\{\left(R_2(2c_2 + R_2)\right)^{1/2} - \left(R_1(R_1 + 2c_1)\right)^{1/2}\right\}(c_1 - c_2)\right.$$

C. 14

$$- \frac{2\left\{(R_2 + C_2) - (R_1 + c_2)\right\}(c_1 R_2 - c_2 R_1)}{\left((R_2 - R_1)(2c_1 + R_1 - 2c_2 - R_2)\right)^{1/2}}\left\{\tan^{-1}\left(\frac{(2c_1 + R_1 - 2c_2 - R_2) R_2}{(R_2 - R_1)(2c_2 + R_2)}\right)^{1/2}\right.$$

$$\left.\left. - \tan^{-1}\left(\frac{(2c_1 + R_1 - 2c_2 - R_2) R_1}{(R_2 - R_1)(2c_1 + R_1)}\right)^{1/2}\right\}\right]$$

Equations C. 11 and C. 12 may be integrated in a similar manner and the following equations obtained

$$x'' = \frac{\Delta y}{c_1 - c_2}\left[\frac{c_1 u_2 - u_1 c_2}{c_1 - c_2} \ln \frac{c_2}{c_1}\left(\frac{c_1 + R_1 - \left(R_1(2c_1 + R_1)\right)^{1/2}}{c_2 + R_2 - \left(R_2(2c_2 + R_2)\right)^{1/2}}\right)\right.$$

$$+ \frac{2(u_1 - u_2)}{c_1 - c_2}\left\{\frac{(c_1 - c_2)^2\left\{c_1(R_2 - u_2) - c_2(R_1 - u_1)\right\} + (u_1 - u_2)^2(c_1 u_2 - c_2 u_1)}{\left\{(R_2 - R_1)(2c_1 + R_1 - 2c_2 - R_2)\right\}^{3/2}}\right\}$$

C. 15

$$X\left\{\tan^{-1}\left(\frac{(2c_1 + R_1 - 2c_2 - R_2) R_2}{(R_2 - R_1)(2c_2 + R_2)}\right)^{1/2} - \tan^{-1}\left(\frac{(2c_1 + R_1 - 2c_2 - R_2) R_1}{(R_2 - R_1)(2c_1 + R_1)}\right)^{1/2}\right.$$

$$\left.\left. - (u_1 - u_2)^2\left\{\frac{\left(R_2(2c_2 + R_2)\right)^{1/2} - \left(R_1(2c_2 + R_1)\right)^{1/2}}{(R_2 - R_1)(2c_1 + R_1 - 2c_2 - R_2)}\right\}\right]$$

C. 16

$$t = \frac{\Delta y}{c_1 - c_2}\left[\ln \frac{c_2}{c_1}\left(\frac{c_1 + R_1 - \left(R_1(2c_1 + R_1)\right)^{1/2}}{c_2 + R_2 - \left(R_2(2c_2 + R_2)\right)^{1/2}}\right)\left(\frac{2(R_2 + C_2 - R_1 - c_1)}{(2c_1 + R_1 - 2c_2 - R_2)(R_2 - R_1)}\right)^{1/2}\right.$$

$$\left. X\left\{\tan^{-1}\left(\frac{(2c_1 + R_1 - 2c_2 - R_2) R_2}{(R_2 - R_1)(2c_2 + R_2)}\right)^{1/2} - \tan^{-1}\left(\frac{(2c_1 + R_1 - 2c_2 - R_2) R_1}{(R_2 - R_1)(2c_1 - R_1)}\right)^{1/2}\right\}\right]$$

The range x within the finite layer is $x = x' + x''$.

The equations C. 14, C. 15 and C. 16 can be considerably simplified if the mean temperature $\bar{c} = \frac{c_1 + c_2}{2}$ is substituted with the change in R remaining the same. Thus equations C. 14, C. 15 and C. 16 become

C. 17
$$x' = \frac{\Delta y \, \bar{c}}{R_2 - R_1} \ln \left(\frac{\bar{c} + R_1 - \left(R_1 (2\bar{c} + R_1)\right)^{1/2}}{\bar{c} + R_2 - \left(R_2 (2\bar{c} + R_2)\right)^{1/2}} \right)$$

C. 18
$$x'' = \bar{u} \, t = \bar{u} \, x' / \bar{c}$$

(Since the motion x'' is small relative to x')

and

C. 19
$$t = \frac{\Delta y}{\bar{c} (R_2 - R_1)} \left\{ \left(R_2 (2\bar{c} + R_2)\right)^{1/2} - \left(R_1 (2\bar{c} + R_1)\right)^{1/2} \right\}.$$

Equations C. 17, C. 18 and C. 19 are the simplified equations of motion of a sound ray in the atmosphere and are most useful — and sufficiently accurate — for fairly rapid computations of sound ray paths.

In the case where $R_1 = R_2$ equations C. 14 through C. 19 are indeterminate since $h = -k$. If $h = -k$, equations C. 10, C. 11 and C. 12 become

$$\frac{dx'}{d\theta} = \frac{R_1 \cos \theta}{h (1 - \cos \theta)^2}$$

$$\frac{dx''}{d\theta} = \frac{c_1 + u_1}{h \cos \theta (1 - \cos \theta)} - \frac{R_1}{h (1 - \cos \theta)^2}$$

$$\frac{dt}{d\theta} = \frac{1}{h \cos \theta (1 - \cos \theta)}$$

and, upon integration and simplification, are expressed as

C. 20 $$x' = \frac{\Delta y\,\bar{c}}{\left(R_1\,(2\bar{c} + R_1)\right)^{1/2}}$$

C. 21 $$x'' = \bar{u}\;x'/\bar{c}$$

C. 22 $$t = \frac{\Delta y\,(R_1 + \bar{c})}{\bar{c}\left(R_1\,(2\bar{c} + R_1)\right)^{1/2}}$$

APPENDIX D

In Figure D-1, Δy represents the altitude at which the sound ray (1) becomes horizontal. The problem is to determine the mean temperature $\bar{c}_2$ and the altitude Δy_2, above Δy_1, for a second ray (2) which will destructively focus with ray (1). If x_1, t_1, and x_2, t_2, represent, respectively, the horizontal distance and time of travel of sound rays (1) and (2) in the altitude interval Δy_1, and if x_3, t_3 represent the horizontal distance and time of travel of sound ray (3) in altitude interval Δy_2, the conditions for destructive focusing (at the ground) of these two rays and expressed by

D. 1

D. 11 $$t_3 = t_1 - t_2$$

D. 12 $$x_3 = x_1 - x_2$$

The equations to be solved then are (See II. 7, II. 8)

D. 2

D. 21 $$t_3 = \frac{\Delta y_2}{-\bar{c}_2 R_1}\left\{-\left(R_1(2\bar{c}+R_1)\right)^{1/2}\right\}$$

D. 22 $$x_3 = \frac{2A - R_1}{2}\left(\frac{\Delta y_2}{-R_2}\right)\ln\left\{\frac{\bar{c}_2 + R_1 - \left(R_1(2c_2+R_1)\right)^{1/2}}{\bar{c}_2}\right\}$$

where A, R_1, and R_2 (=0) are determined for a particular ray in a particular atmosphere (See Appendix C).

From equation D. 21

D. 3

$$\frac{-\bar{c}_2}{-\left\{R_1(\bar{2}c_2+R_1)\right\}^{1/2}} = \frac{\Delta y_2^2}{t_3 R_1}$$

$$\frac{\bar{c}_2^2}{2\bar{c}_2+R_1} = \frac{\Delta y_2^2}{t_3^2 R_1}$$

$$\bar{c}_2 = \frac{\Delta y_2^2}{t_3^2 R_1} + \frac{\Delta y_2}{t_3}\sqrt{\frac{\Delta y_2^2}{t_3^2 R_1^2} + 1}$$

GEOMETRICAL EXPRESSION OF PROBLEM IN APPENDIX D

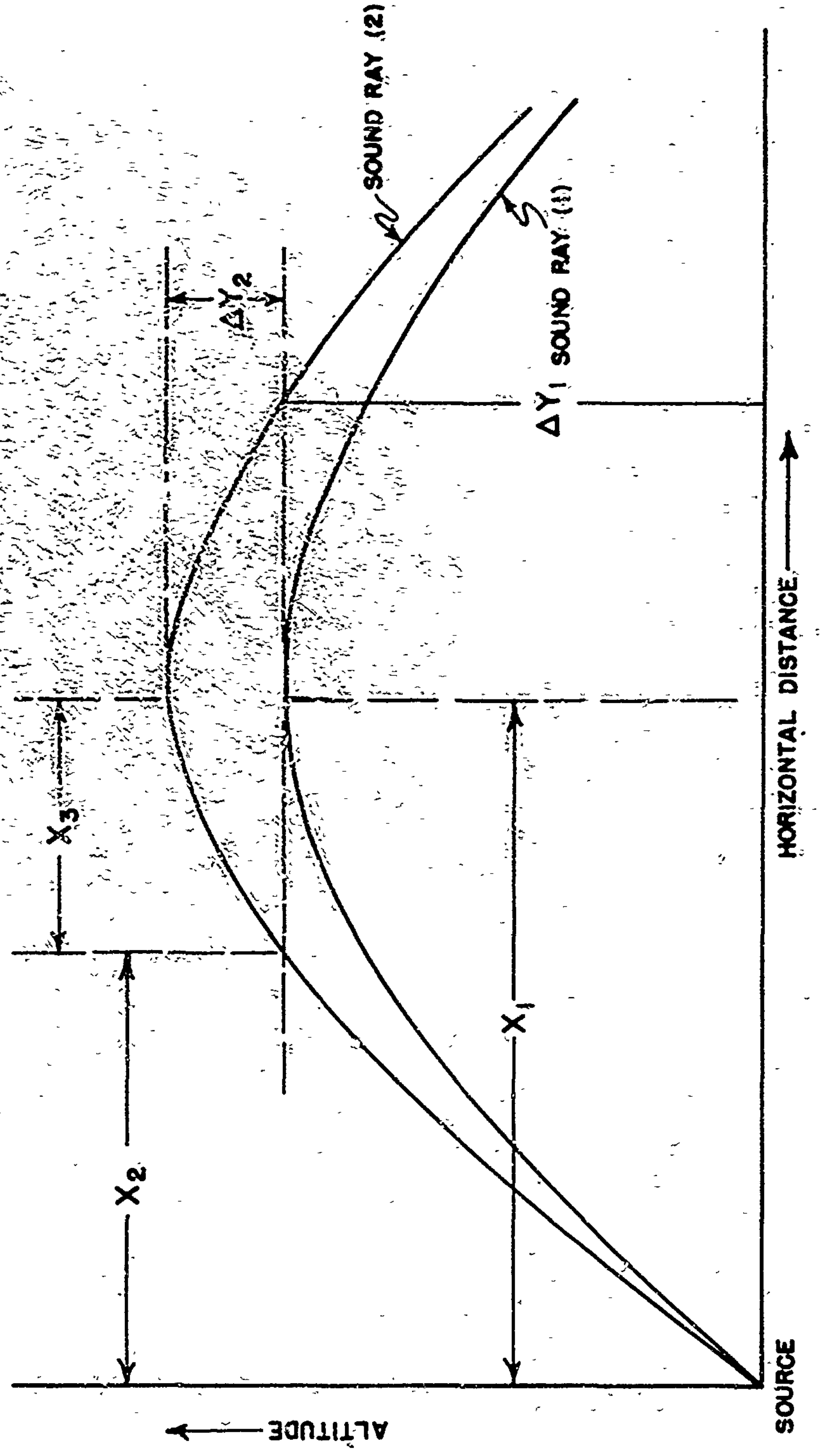

FIGURE D-1

From Equation D. 22

$$\frac{-2R_1 x_3}{(2A - R_1)(\Delta y_2)} = \ln \left\{ \frac{\bar{c}_2 + R_1 - \left(R_1 (2\bar{c}_2 + R_1)\right)^{1/2}}{\bar{c}_2} \right\}$$

$$e^{\frac{-2R_1 x_3}{(2A - R_1)(\Delta y_2)}} = \frac{\bar{c}_2 + R_1 - \left\{R_1 (2\bar{c}_2 + R_1)\right\}^{1/2}}{\bar{c}_2} = e^{\frac{-m}{\Delta y_2}}$$

where

$$m = + \frac{2R_1 x_3}{2A - R_1}$$

Continuing;

$$1 + \frac{R_1}{\bar{c}_2} - \frac{\sqrt{R_1 (2\bar{c}_2 + R_1)}}{\bar{c}_2} = e^{\frac{-m}{\Delta y_2}}$$

$$\frac{\sqrt{R_1 (2\bar{c}_2 + R_1)}}{\bar{c}_2} = e^{\frac{-m}{\Delta y_2}} - \frac{R_1}{\bar{c}_2} - 1$$

Squaring the above expressions one obtains

$$\frac{2(R_1 \bar{c}_2) + R_1^2}{\bar{c}_2^2} = e^{\frac{-2m}{\Delta y_2}} - 2\frac{R_1}{\bar{c}_2} e^{\frac{-m}{\Delta y_2}} - 2e^{\frac{-m}{\Delta y_2}} + \frac{R_1^2}{\bar{c}_2^2} + \frac{2R_1}{\bar{c}_2} + 1$$

$$2\frac{R_1}{\bar{c}_2} e^{\frac{-m}{\Delta y_2}} = e^{\frac{-2m}{\Delta y_2}} - 2e^{\frac{-m}{\Delta y_2}} + 1 = \left(e^{\frac{-m}{\Delta y_2}} - 1\right)^2$$

whence,

$$\frac{2R_1}{\bar{c}_2} = \frac{\left(e^{\frac{-m}{\Delta y_2}} - 1\right)^2}{e^{-\frac{m}{\Delta y_2}}} = e^{\frac{-m}{\Delta y_2}} - 2 + e^{\frac{m}{\Delta y_2}}$$

and,

$$\frac{R_1}{\bar{c}_2} = \frac{e^{\frac{-m}{\Delta y_2}} + e^{\frac{m}{\Delta y_2}}}{2} - 1 = \cos h \frac{m}{\Delta y_2} - 1$$

and,

D. 4

$$\bar{c}_2 = \frac{R_1}{\cos h \frac{m}{\Delta y_2} - 1}$$

Now equating D. 4 and D. 3 and setting $t_3 R_1 = n$ results in

$$\frac{R_1}{\cos h \frac{m}{\Delta y_2} - 1} = \frac{\Delta y_2}{t_3}\left[\frac{\Delta y_2}{n} + \sqrt{\frac{\Delta y_2^2}{n^2} + 1}\right]$$

$$\frac{1}{\cos h \frac{m}{\Delta y_2} - 1} = \frac{\Delta y_2}{n}\left[\frac{\Delta y_2}{n} + \sqrt{\frac{\Delta y_2^2}{n^2} + 1}\right]$$

or;

$$\frac{1}{\cos h \frac{m}{\Delta y_2} - 1} = \frac{\Delta y_2^2 + n \Delta y_2^2 \sqrt{\frac{\Delta y_2^2}{n^2} + 1}}{n^2}$$

Now, multiplying the numerator and denominator of the right side of the equation above by the factor $\left(\Delta y_2^2 - n\,\Delta y_2 \sqrt{\frac{\Delta y_2^2}{n^2} + 1}\right)$ yields

$$\frac{1}{\cos h \frac{m}{\Delta y_2} - 1} = \frac{\Delta y_2^4 - n^2 \Delta y_2^2\left(\frac{\Delta y_2^2}{n^2} + 1\right)}{n^2\left(\Delta y_2^2 - n \Delta y_2 \sqrt{\frac{\Delta y^2}{n^2} + 1}\right)}$$

or, after simplification

$$\cos h \frac{m}{\Delta y_2} - 1 = -1 + \sqrt{1 + \frac{n^2}{\Delta y_2^2}}$$

$$\cos h^2 \frac{m}{\Delta y_2} = 1 + \frac{n^2}{\Delta y_2^2}$$

$$\cos h^2 \frac{m}{\Delta y_2} - 1 = \sin h^2 \frac{m}{\Delta y_2} = \frac{n^2}{\Delta y_2^2}$$

$$\sin h \frac{m}{\Delta y_2} = \frac{n}{\Delta y_2}$$

or

$$\frac{\sin h \frac{m}{\Delta y_2}}{\frac{m}{\Delta y_2}} = \frac{n}{m}$$

and, from the series expression for $\sin h \frac{m}{\Delta y_2}$,

D. 5 $$\Delta y_2 = m \sqrt{6\left(\frac{m}{n} - 1\right)}$$ very closely,

Substitution of D. 5 into D. 4 yields

D. 6 $$\bar{c}_2 = \frac{R_1}{\cos h \frac{1}{\sqrt{6\left(\frac{m}{n} - 1\right)}} - 1}$$

Equations D. 5 and D. 6 define the atmospheric structure necessary above a structure of altitude Δy_1, which will destructively focus the two rays of sound (1) and (2) defined and discussed above.

APPENDIX E

Summary of calculations for the three atmospheric structures studied in the text.

The equations used in determining the path of a sound ray through the atmosphere (see appendix C) are given as

E. 1
$$x' = \frac{\bar{c}\,\Delta y}{R_2 - R_1} \ln \left\{ \frac{\bar{c} + R_1 - \left(R_1 (2\bar{c} + R_1)\right)^{1/2}}{\bar{c} + R_2 - \left(R_2 (2\bar{c} + R_2)\right)^{1/2}} \right\}$$

E. 2
$$x'' = x' \cdot \frac{\bar{u}}{\bar{c}}$$

E. 3
$$t = \frac{\Delta y}{\bar{c}\left(R_2 - R_1\right)} \left\{ \left(R_2 (2\bar{c} + R_2)\right)^{1/2} - \left(R_1 (2\bar{c} + R_1)\right)^{1/2} \right\}$$

Now, since the total horizontal range x of a sound ray in a segment of the atmosphere, Δy, is given by $x = x' + x''$, equations E. 1 and E. 2 may be combined to give

E. 4
$$x = \Delta y \left[\frac{A}{R_1 + R_2} - 1/2 \right] \ln \left\{ \frac{\bar{c} + R_1 - \left(R_1 (2\bar{c} + R_1)\right)^{1/2}}{\bar{c} + R_2 - \left(R_2 (2\bar{c} + R_2)\right)^{1/2}} \right\}$$

The investigation procedure for examining the three idealized atmospheres (Figure 7) was explained adequately in the text and only the results will be given in this appendix.

1) Example (a) Figure 7 a.

For this example an isothermal structure was considered and an altitude interval Δy_1, chosen so that a sound ray of initial angle $\theta = 2^\circ$ (angle with respect to horizontal) would be just refracted in the interval. The horizontal range and time of passage of the sound ray $\theta = 2^\circ$ in the layer Δy_1, was then calculated by means of equations E. 3 and E. 4. The horizontal ranges and times of passage of sound rays $\theta = 4^\circ$, $\theta = 6^\circ$, $\theta = 8^\circ$ and $\theta = 10^\circ$ were calculated for the same altitude interval Δy_1. Table I gives the values for x and t of sound rays $\theta = 2^\circ$, $\theta = 4^\circ$, $\theta = 6^\circ$, $\theta = 8^\circ$ and $\theta = 10^\circ$ in the altitude interval Δy_1. In addition the various constants for use in equations E. 3 and E. 4 are tabulated.

TABLE I

			Ft. sec.	Ft/sec.						Ft.	Sec.	Ft.	Sec.	Ft.
θ	C_1	C_2	C	u_1	u_2	$\bar{u}$	A	R_1	R_2	Y_1	t	X	t (2) -t (0)	X (2)
2°	1100	1100	1100	-10	-9.34	-9.67	1090.66	0.66	0	165	8.660	9445.5	0	0
4°	1100	1100	1100	-10	-9.34	-9.67	1092.64	2.64	1.98	165	2.324	2532.3	6.336	6913.2
6°	1100	1100	1100	-10	-9.34	-9.67	1096.38	6.38	5.72	165	1.429	1548.3	7.231	7897.2
8°	1100	1100	1100	-10	-9.34	-9.67	1100.78	10.78	10.12	165	1.096	1185.7	7.564	8259.8
10°	1100	1100	1100	-10	-9.34	-9.67	1106.94	16.94	16.26	165	0.873	937.7	7.787	8507.8
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)

The equations for computing A and R are given by

$$c \sec \theta + u = A,$$

$$A - (c + u) = R.$$

In order that destructive focusing is to occur between $\theta = 2^\circ$ and $\theta = 4^\circ$, $\theta = 6^\circ$, $\theta = 8^\circ$ and $\theta = 10^\circ$, it is obvious that the time of travel and the horizontal range for rays $\theta = 4^\circ$, 6°, 8° and 10° in an altitude Δy_2 above Δy_1 must be equal to the time and distance differences (columns (14) and (15) of Table I) between $\theta = 2^\circ$ and the other sound rays. Substituting these time and distance differences into equations E. 3 and E. 4 enables one to solve for the atmospheric structure of altitude interval Δy_2. Table II tabulates the characteristics of the atmosphere above Δy_1 which will produce destructive focusing.

TABLE II

	Ft/sec.			Ft/sec		Ft.			
θ	$\bar{C}$	C_1	C_2	u_1	u_2	Δy_2	R_1	R_2	A
4°	1100.15	1100	1100.30	-9.34	- 7.66	209.0	1.98	0	1092.64
6°	1106.4	1100	1112.8	-9.34	-15.4	406.2	5.72	0	1096.38
8°	1109.2	1100	1118.4	-9.34	-17.6	565.4	10.12	0	1100.78
10°	1114.6	1100	1129.2	-9.34	-22.3	739.0	16.28	0	1106.94

Table II illustrates quite clearly the temperature function necessary for destructive focusing.

2) Example (b) Figure 7b.

In example (b) a slightly different procedure was followed. Three sound rays, $\theta = 2^\circ$, $\theta = 4^\circ$, and $\theta = 6^\circ$, were traced through an altitude interval Δy_1 which was allowed to vary but within which the mean temperature was held constant. Effectively this means that three separate atmospheres were studied (temperature and wind gradients varied over Δy_1). However, the wind gradients were so chosen that the

sound ray $\theta = 2^\circ$ was just refracted in the altitude interval Δy_1. Equations E. 3 and E. 4 were then used to determine the temperature structure and altitude interval Δy_2 which would destructively focus sound rays $\theta = 4^\circ$ and $\theta = 6^\circ$ with the ray $\theta = 2^\circ$. The results of these calculations are given in Table III below.

TABLE III

	Ft/sec				Ft/sec			Ft.		
θ	C_1	C_2	C_3	$\bar{C}_1$	u_1	u_2	u_3	Δy_1	Δy_2	$\bar{C}_2$
2°	1100	1110	--	1105	10	0.66	--	200	0	--
4°	1100	1110	1101	1105	10	0.66	11.64	200	253.4	1105.5
6°	1100	1110	1109.9	1105	10	0.66	6.48	200	491.7	1109.85
4°	1100	1110	1103.8	1105	10	0.66	8.84	400	507.2	1106.9
6°	1100	1110	1109.9	1105	10	0.66	6.48	400	983.3	1109.85
4°	1100	1110	1103.8	1105	10	0.66	8.84	600	760.8	1106.9
6°	1100	1110	1109.9	1105	10	0.66	6.48	600	1474.9	1109.85

C_1, C_2, u_1 and u_2 represent the sound velocity and wind velocity at the lower and upper limits, respectively, of Δy_1; C_2, C_3, u_2 and u_3 represent the sound velocity at the lower and upper limits of the altitude interval y_2

$$\bar{C}_1 = \text{mean temperature through } \Delta y_1$$

$$\bar{C}_2 = \text{mean temperature through } \Delta y_2$$

Although Table III indicates that a negative temperature gradient exists in Δy_2 for the atmospheric models of example (b), the positive temperature gradient which has been postulated as a necessary condition for destructive focusing appears in the altitude interval Δy_1.

3) Example (c) Figure 7c.

A dry adiabatic temperature gradient was assumed to extend from the ground to 1000 ft. (-5.4°F/1000 ft.) Further, a wind velocity gradient of + 10 ft/sec., with an initial velocity $u_1 = 0$, was assumed for the same altitude interval. It was found that a sound ray of initial angle $\theta = 5.232^\circ$ would be just refracted in the 1000 ft. interval. Therefore sound rays of initial angles $\theta = 6^\circ$ and $\theta = 7^\circ$ were investigated to determine the atmospheric conditions above 1000 ft. necessary to destructively focus these rays with ray $\theta = 5.232^\circ$.

As might be expected from the results already discussed in the two preceding examples, the mean temperature in altitude interval Y_2 for rays $\theta = 6^\circ$ and $\theta = 7^\circ$ had to be greater than the mean temperature in Δy_1 and this specified a positive temperature gradient of no mean proportions in the altitude interval Δy_2 (dependent upon the sound ray investigated and greater than 5.4° F/1000 ft. for $\theta = 6^\circ$). Thus it is indicated that in the normal type atmospheric structure where the temperature decreases with altitude, a considerable temperature inversion is necessary to produce destructive focusing of sound rays.